Coding for the $\ell_\infty$-limited Permutation Channel

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Coding for Synchronization Errors

- Symbols received in incorrect order
- Channel action characterized by permutation: $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5 = (4,1,3,2,5)$
- Channel action: $y_i = x_{\pi_i}$
Permutation Channels

\[ \begin{array}{ccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 \\
\end{array} \quad \rightarrow \quad \begin{array}{ccccc}
  x_4 & x_1 & x_3 & x_2 & x_5 \\
\end{array} \]

\[ \pi = (4,1,3,2,5) \]

Has been previous work on permutation channels:

- Differ on assumptions regarding possible \( \pi \)
- Stochastic (channel induces \( \pi \) governed by dist.)
- Worst-case (channel maliciously chooses worst \( \pi \) from set of permissible permutations)
- Differ in input (output) alphabet \( \Sigma \)
Permutation Channels

Most related previous work:
• “Bit-shift magnetic recording channel” (RLL + errors)
  • $\Sigma = \{0,1\}$
  • Limitation on $\pi$: $|i-\pi_i| \leq r$ for all $x_i=1$
  • Worst case: over such $\pi$ [KolesnikKrachkovsky][Ytrehus][Krachkovsky][Abdel-GhaffarWeber] ...
  • Stochastic: over such $\pi$ [ShamaiZehavi]

More ...
• “Trapdoor channels”: [BlackwellBreimanThomasian][Benjamin][AhlswedeKaspi][Chan][Kobayashi][Piret][KobayashiMorita][PermuterCuffRoyWeissman]
• Context of “rank modulation”, e.g. when the codewords themselves are permutations: [KløveLinTsaiTzeng][TamoSchwartz][ZhouSchwartzJiangBruck][Kløve][Schwartz][WangMazumdarWornell][Lehmer]
• Other models: [KovacevicVukobratovic][WalshWeber][KovacevicPopovski]
Our Model and Results

- **Model** \((q, r)\):
  - \(|\Sigma| = q\)
  - Limitation on \(\pi\): bounded \(l_\infty\)-distance = \(\max |i - \pi_i| \leq r\)
  - Worst case (combinatorial) analysis

- **Results**:
  - Set out to perform a comprehensive study
  - Error-sphere size (depends on codeword), average size
  - Upper bounds
  - Code construction (lower bounds)
  - Open Problems
Warm Up

\[ \Sigma = \{0,1\} \text{ and } \max_i |i-\pi_i| \leq 1 \text{ and } n=3 \]

- **Observation**: weight of codeword does not change
- **Can communicate 4 codewords**: 000, 001, 011, 111
- **Rate**: \( \frac{2}{3} \)

- **Observation**: there exists a covering code of size 4
  - 000, 010, 101, 111
  - *Covering code acts as upper bound*
- **Thus for** \( n=3 \), rate is **2/3**
Larger n?

- **Observation**: Rate R code for $n=3$ does not necessarily imply a rate R code for larger $n$.
- **Number of “bit errors”** ($x_i \neq y_i$) increases as $n$ increases.
- **Main Goal**: What is the rate for large $n$?

- **Major difficulty**: error-sphere sizes
  - $\sum = \{0,1\}$ and $\max_{i} |i-\pi_i| \leq 1$.
  - $x_1 x_2 x_3 x_4$

- **GV type lower bound**
- **Naïve upper bounds**
Background

• For two permutations $\pi, \sigma \in S_n$, the $l_\infty$-distance is
  \[ d_\infty(\pi, \sigma) = \max_i |\pi(i) - \sigma(i)| \]
  \[ wt_\infty(\pi) = d_\infty(\pi, \text{Id}) \]

• The ball of radius $r$ centered in $x \in \Sigma^n$ is
  \[ B_r(x) = \{ \pi x \mid \pi \in S_n, wt_\infty(\pi) \leq r \} \]

• $C \subseteq \Sigma^n$ is an $r$-ECC if for all $c_1, c_2 \in C$
  \[ B_r(c_1) \cap B_r(c_2) = \emptyset \]

• $A_q(n; r) = \text{largest } M \text{ s.t. there exists a length-} n \text{ r-ECC of size } M$

• $C \subseteq \Sigma^n$ is an $R$-covering code if
  \[ \bigcup_{c \in C} B_R(c) = \Sigma^n \]

• $K_q(n; R) = \text{smallest } M \text{ s.t. there exists a length-} n \text{ R-covering code of size } M$
Basic Properties

• The $\ell_\infty$-distance between $x,y \in \Sigma^n$, $d_\infty(x,y)$, is the min $w$ s.t. there exists $\pi \in S_n$, $wt_\infty(\pi)=w$ and $y=\pi x$
  – If such $\pi$ doesn’t exist then the distance is $\infty$
• $n_a(x) = |\{i \in [n] : x_i=a\}|$
• $x,y \in \Sigma^n$ have equal composition if $n_a(x)=n_a(y)$ for all $a \in \Sigma$
• $d_\infty(x,y) = \infty$ iff $x$ and $y$ have different compositions
• **Lemma**: If $X \subseteq \Sigma^n$ is a set of strings with equal composition, then the $\ell_\infty$-distance defines a **metric** over $X$
• **Corollary**: $C$ is an r-ECC if $d_\infty(c_1,c_2) \geq 2r+1$ for all $c_1,c_2 \in C$
Basic Problems

• **Problem 1**: How to find $d_\infty(x, y)$, for $x, y \in \Sigma^n$?

• **Answer**: $d_\infty(x, y) = \max_{a \in \Sigma, j \in [n_a(x)]} |L_a(j; x) - L_a(j; y)|$

  finding $\pi \in S_n$, such that $y = \pi x$ can be done in $O(n)$ time

  – $L_a(j; x)$ = the $j^{th}$ occurrence of $a$ in $x$

• **Example**:

  – $x = 00011$, $y = 01001$, $\Sigma = \{0, 1\}$
  – $L_0(1; x) = 1$, $L_0(2; x) = 2$, $L_0(3; x) = 3$, $L_1(1; x) = 4$, $L_1(2; x) = 5$
  – $L_0(1; x) = 1$, $L_0(2; x) = 3$, $L_0(3; x) = 4$, $L_1(1; x) = 2$, $L_1(2; x) = 5$
  – $\max_{a \in \Sigma, j \in [n_a(x)]} |L_a(j; x) - L_a(j; y)| = 2$
  – $\pi = 13425$, $\text{wt}_\infty(\pi) = 2$
Basic Problems

• **Problem 2:** How to find $|B_r(x)|$?
  
  $B_r(x) = \{ \pi x \mid \pi \in S_n \text{ wt}_\infty(\pi) \leq r \}$
  
  - $B_1(000) = \{000\}, B_1(111) = \{111\}$
  - $B_1(010) = \{100, 010, 001\}, B_1(101) = \{011, 101, 110\}$

• Exact sphere size depends on “run” structure of the center

• An antirun is a (maximal) sequence of different symbols
  
  - $011010001101 \Rightarrow 01.1010.001$.

• **Answer:** Let $P(x) = (r_1, r_2, \ldots, r_k)$ be the antirun profile, then $|B_1(x)| = \prod F_{r_i}$, where $F_{r_i}$ is the $r_i$'th Fibonacci number
  
  - $P(0.0.0) = (1,1,1), |B_1(000)| = F_1 F_1 F_1 = 1$
  - $P(01.10.0) = (2,2,1), |B_1(01100)| = F_2 F_2 F_1 = 4$
    
    $B_1(01100) = \{01100, 10100, 01010, 10010\}$
  
  - $P(0101) = (4), |B_1(0101)| = F_4 = 5$
    
    $B_1(0101) = \{0101, 1001, 1010, 0011, 0110\}$
Basic Problems

• **Problem 2**: How to find $|B_r(x)|$?
• Exact sphere size depends on “run” structure of the center
• An antirun is a (maximal) sequence of different symbols
• **Answer**: Let $P(x) = (r_1, r_2, \ldots, r_k)$ be the antirun profile, then
  
  $|B_1(x)| = \prod F_{r_i}$, where $F_{r_i}$ is the $r_i$'th Fibonacci number

• $\min_{x \in \{0,1\}^n} |B_1(x)| = 1$
• $\max_{x \in \{0,1\}^n} |B_1(x)| \leq F_n$ since $|B_r(x)| \leq |\{\pi \in S_n \mid wt_\infty(\pi) \leq r\}|$
  and $|\{\pi \in S_n \mid wt_\infty(\pi) \leq 1\}| = F_n$

• $\max_{x \in \{0,1\}^n} |B_1(x)| = |B_1(0101\ldots01)| = F_n$ since $F_{a+b} > F_a F_b$
Basic Problems

• **Problem 2**: How to find $|B_r(x)|$?
• Exact sphere size depends on “run” structure of the center
• An antirun is a (maximal) sequence of different symbols
• **Answer**: Let $P(x) = (r_1, r_2, ..., r_k)$ be the antirun profile, then
  $$|B_1(x)| = \prod F_{r_i},$$
  where $F_{r_i}$ is the $r_i$'th Fibonacci number
• $\min_{x \in \{0, 1\}^n} |B_1(x)| = 1; \max_{x \in \{0, 1\}^n} |B_1(x)| = F_n$
• $B_{r,q,n} = (1/q^n) \cdot \sum_{x \in \mathbb{Z}^n} |B_r(x)|$
• $B_{1,q,n} = B_{1,q,n-1} + (q-1)/q \cdot B_{1,q,n-2}$
• $B_{1,q,n} = \left( \frac{\ell + \sqrt{\ell}}{2\ell} \right)^n \left( \frac{1 + \sqrt{\ell}}{2} \right)^n + \left( \frac{\ell - \sqrt{\ell}}{2\ell} \right)^n \left( \frac{1 - \sqrt{\ell}}{2} \right)^n ; \ell = 5 - 4/q$
• $B_{1,2,n} \approx 0.789 \cdot \left( \frac{1 + \sqrt{3}}{2} \right)^n$
• Can use the GV to get a lower bound on the code cardinalities
• However... we know only how to calculate $B_{1,q,n}$
• **Open Problem**: Find $|B_r(x)|$ for all $x$ and $r$ and $B_{r,q,n}$
Upper Bounds

- **Theorem**: For all $n$ and $r$
  
  $$A_q(n;r) \leq K_q(n;r) \leq \left( \frac{r + q}{q - 1} \right)^{n/(r+1)}$$

- For $r=1$:
  
  $$A_q(n;1) \leq K_q(n;1) \leq \left[ q + 2 \left( \begin{array}{c} q \\ 2 \end{array} \right) + 2 \left( \begin{array}{c} q \\ 3 \end{array} \right) \right]^{n/3}$$
  
  - $A_2(n;1) \leq 4^{n/3}$
  - Results from the covering code as an upper bound

- **Problem**: Can we get a better upper bound?

- **Answer**: Yes, using a generalized sphere packing bound
The Sphere Packing Bound

• Upper bound on a code $C$ with min dist $2r+1$

$$|C| \leq \frac{2^n}{B(r)} - B(r) = \sum_{i=0}^{r} \binom{n}{i}$$

• This bound is valid for other cases as well where the error graph is regular ($|X|/\Delta_r$)

• Q: what happens if the graph is not regular?
The Deletion Channel

• An example of non-regular graph
  – 10010 → 0010, 1010, 1000, 1001
  – 11100 → 1100, 1110
  – 10101 → 0101, 1101, 1001, 1011, 1010

• It is not possible to apply the sphere packing bound 😞

• Previous results
  – Levenshtein ’66: asymptotic upper bound
  – Kulkarni & Kiyavash ’12: a method to derive explicit non-asymptotic upper bound using tools from hypergraph theory
Hypergraphs

- Let $H=(X,E)$ be a hypergraph, where
  - $X=\{x_1,\ldots,x_n\}$ - set of vertices, $E=\{E_1,\ldots,E_m\}$ - set of hyperedges
  - $A$ is a binary $n \times m$ incidence matrix of $H$

- **Matching** - a collection of pairwise disjoint hyperedges
  - The matching number $\nu(H)$ - size of the largest matching
    \[
    \nu(H) = \max \left\{ \sum_{i=1}^{m} z_i : A \cdot z \leq 1, z \in \{0,1\}^m \right\}
    \]

- **Transversal** - a vertices subset that intersects every hyperedge
  - The transversal number $\tau(H)$ - size of the smallest transversal
    \[
    \tau(H) = \min \left\{ \sum_{i=1}^{n} w_i : A^T \cdot w \geq 1, w \in \{0,1\}^n \right\}
    \]

- These problems satisfy weak duality $\nu(H) \leq \tau(H)$
Hypergraphs

- The matching number \( \nu(H) = \max \left\{ \sum_{i=1}^{m} z_i : A \cdot z \leq 1, z \in \{0,1\}^m \right\} \)

- The transversal number \( \tau(H) = \min \left\{ \sum_{i=1}^{n} w_i : A^T \cdot w \geq 1, w \in \{0,1\}^n \right\} \)

- These problems satisfy weak duality \( \nu(H) \leq \tau(H) \)

- The relaxation versions of these problems satisfy strong duality

\[
\tau^*(H) = \min \left\{ \sum_{i=1}^{n} w_i : A^T \cdot w \geq 1, w \in \mathbb{R}_+^n \right\}
\]

\[
\nu^*(H) = \max \left\{ \sum_{i=1}^{m} z_i : A \cdot z \leq 1, z \in \mathbb{R}_+^m \right\}
\]

satisfy strong duality

\[
\nu(H) \leq \nu^*(H) = \tau^*(H) \leq \tau(H)
\]

- Every vector \( w \) in \( \tau^*(H) \) is called a fractional transversal
The Deletion Channel – KK’12

• Define a hypergraph \( H(X,E) \):
  – \( X = \{0,1\}^{n-1} \), \( E = \{\text{all } 2^n \text{ single-deletion balls}\} \)

• Every single-deletion correcting code of length \( n \) is a matching in the hypergraph \( H \)

• Find the value of \( \tau^*(H) \) or any fractional transversal to get an explicit upper bound

\[
\nu(H) \leq \nu^*(H) = \tau^*(H) \leq \tau(H)
\]

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The General Case

- \( G=(X,E) \) is a graph describing an error channel graph
  - \( X \) = the set of all possible words (transmitted and received)
  - \( E \) = the set of vertices pairs of dist one
    - The distance \( d(x,y) \) b/w \( x \) and \( y \) is the length of the shortest path from \( x \) to \( y \) (not necessarily symmetric)
    - \( B_r(x) = \{ y \in X : d(x,y) \leq r \} \); \( \deg_r(x) = |B_r(x)| \)
- For any \( r>0 \), \( H(G,r)=(X_r,E_r) \) is a hypergraph for \( G \)
  - \( X_r=X \), \( E_r=\{B_r(x) : x \in X \} \)
- Every \( r \)-ECC \( C \) in \( G \) is a matching in \( H(G,r) \)
- \( A_G(n,r) \) - the max size of a length-\( n \) \( r \)-ECC in \( G \)

For every \( r>0 \):
\[
A_G(n,r) \leq \tau^*(H(G,r))
\]

The Generalized Sphere Packing Bound (GSPB)
GSPB for the $\ell_\infty$-metric

- $G=(X_{n,w}, E_{n,w})$
  - $X_{n,w} =$ length-$n$ binary vectors of weight $w$
  - $E_{n,w} =$ all radius-1 balls centered in $x \in X_{n,w}$

- **Open**: Finding a closed formula for the upper bound

- Numerical results for the linear programming problem give the following values

- **Observation**: The “average” sphere packing bound is NOT a valid upper bound in this channel
  - The sphere size is $0.789 \cdot \left( \frac{1 + \sqrt{3}}{2} \right)^n$
  - If it were a valid upper bound, then 0.55 will be an upper bound on the rate, which does not hold (will see later)
Code Constructions

- **Recall**: for \( n=3 \) one can achieve optimal rate of \( 2/3 \)
- **Goal**: to obtain optimal constructions for large \( n \)
- **Direct construction (constrained coding)**:
  - Used in study of bit-shift magnetic recording channel
    [ShamaiZehavi][Krachkovsky][KolesnikKrachkovsky][Ytrehus][Krachkovsky] [Abdel-GhaffarWeber] ...
  - **Idea**: Identify blocks that can be decoded sequentially
  - **More formally**: Given a set of blocks \( B \subseteq \Sigma^* \), let
    \[ C_n(B) = \{b_1b_2\cdots b_m \mid b_1, b_2, \ldots, b_m \in B, \Sigma |b_i| = n \} \]
  - The asymptotic rate is given by
    \[ \log_2 \lambda = \limsup_{n \to \infty} \frac{\log_2 |C_n(B)|}{n} \]
    \( \lambda \) is the largest solution of the equation
    \[ \sum_{b \in B} x^{-|b|} = 1 \]
Code constructions

- **Direct construction (constrained coding):**
  - *Idea:* Identify blocks that can be decoded sequentially
  - **More formally:** Given a set of blocks $\mathcal{B} \subseteq \Sigma^*$, let
    $$C_n(\mathcal{B}) = \{b_1 b_2 \cdots b_m \mid b_1, b_2, \ldots, b_m \in \mathcal{B}, \Sigma |b_i| = n\}$$
  - The asymptotic rate is given by
    $$\log_2 \lambda = \limsup_{n \to \infty} \frac{\log_2 |C_n(\mathcal{B})|}{n}$$
    $\lambda$ is the largest solution of
    $$\sum_{\mathcal{B}} x^{-|b|} = 1$$
  - The set $\mathcal{B} = \{0^3 i 1 \mid 0 \leq i \}$ was used to generate a code which satisfies the RLL constraint with asymptotic rate 0.551 [Krachkovsky]
Code constructions

• **Direct construction (constrained coding):**
  - **Idea:** Identify blocks that can be decoded sequentially
  - **More formally:** Given a set of blocks $B \subseteq \Sigma^*$, let $C_n(B) = \{b_1b_2\cdots b_m | b_1, b_2, \ldots, b_m \in B, \Sigma |b_i| = n\}$
  - The asymptotic rate is given by $\log_2 \lambda = \limsup \frac{\log_2 |C_n(B)|}{n}$
    - $\lambda$ is the largest solution of $\sum_{b \in B} x^{-|b|} = 1$ as $n \to \infty$
  - The set $B = \{0^3i1 | 0 \leq i \}$ was used to generate a code which satisfies the RLL constraint w/ asymp rate 0.551

• **Theorem:** The code $C_n(B)$ for $B = B_1UB_2UB_3UB_4$ is an ECC which allows decoding in time $\Theta(n)$. The asymptotic rate is $\log_2 \lambda = 0.5875$, where $\lambda$ the largest solution of $x^7-3x^4-2=0$

  $B_1 = \{0^{2+3i}1 | i \geq 0\}$, $B_2 = \{0^{3+3i}1^4 | i \geq 0\}$
  $B_3 = \{1^{2+3i}0 | i \geq 0\}$, $B_4 = \{1^{3+3i}0^4 | i \geq 0\}$
Code constructions

• **Recursive construction:**
  - The inner code: for each $a \in \mathbb{Z}_q$, there exists a single-ECC $C_a$, s.t. for each $c \in C_a$, $\text{wt}_q(c) = a$
  - The outer code: $C' \in \mathbb{Z}_q^k$ - a set of vectors with distinct $q$-weight
  - Construct the code $C = \bigcup_{(a_1,\ldots,a_k) \in C'} C_{a_1} \times \cdots \times C_{a_k}$

• **Theorem:** The code $C$ is a single ECC of size
  $$\sum_{(a_1,\ldots,a_k) \in C'} \prod_{i=1}^k |C_{a_i}|$$

• **Example:**
  - $C_0 = \{000,110\}$, $C_1 = \{100,111\}$, $C' = \{0101\ldots01,1010\ldots10\}$
  - We get a single ECC of length $3k$ and cardinality $2 \cdot 2^k$ (rate = 1/3)

• **Computer search:** a code of length 24 and rate 0.65

• Using this construction, we get codes of arbitrary length and rate **0.609**, which is our best asymptotic result
Summary: Upper & Lower Bounds

- Best asymptotic construction achieves rate 0.609
- Best (long) fixed length rate is 0.65 for \( n = 24 \)
- Upper bound: 2/3
- **Open Problem**: Find codes with asymptotic rate > 0.609
- **Conjecture**: The best asymptotic rate is 2/3
- **Other problems**:
  - Extensions of the results and constructions for larger radii
  - Study the capacity in case there is a fraction of \( p \) transpositions
  - Study the problem for other families of permutations, e.g. Ulam, Kendall’s tau etc.

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