Optimal Fractional Repetition Codes for Distributed Storage Systems

Natalia Silberstein

Joint work with Tuvi Etzion

Technion
Israel Institute of Technology
Distributed storage system (DSS)

Source

Google
Facebook
Amazon
Dropbox

storage node 1
storage node 2
storage node n
Distributed storage system (DSS)

- Failures are the norm rather than the exception

Google
Facebook
Amazon
Dropbox

source

storage node 1

storage node 2

storage node n

data collector
Distributed storage system (DSS)

- **Failures** are the norm rather than the exception

- **Redundancy** for reliability
  - Replication
  - Erasure coding

source

storage node 1

storage node 2

storage node n
Coding for distributed storage system

- **Erasure codes:**
  Using \((n, k)\) erasure code:
  - Partition the data into \(k\) blocks
  - Generate \(n\) blocks, store each block in a new node

\((n, k)\) **MDS property:**
Reconstruct the data from any \(k\) nodes
Coding for distributed storage system

- **Erasure codes:**
  - Using \((n, k)\) erasure code:
    - Partition the data into \(k\) blocks
    - Generate \(n\) blocks, store each block in a new node
  - \((n, k)\) MDS property:
    - Reconstruct the data from any \(k\) nodes
- **Regenerating codes**
  - Efficient node repairs
Regenerating codes (*)

- Minimum storage regenerating (MSR) codes
- Minimum bandwidth regenerating (MBR) codes

DRESS codes(*)
(Distributed Replication based Exact Simple Storage)

• Minimum repair bandwidth (like MBR)
• Uncoded repair (repair by transfer)
• Table based repairs (specific $d$)

(*) S. El Rouayheb and K. Ramchandran, “Fractional repetition codes for repair in distributed storage systems”, 2010
DRESS codes(*)
(Distributed Replication based Exact Simple Storage)

• Minimum repair bandwidth (like MBR)
• Uncoded repair (repair by transfer)
• Table based repairs (specific $d$)

Allow to store more data than MBR codes!

(*) S. El Rouayheb and K. Ramchandran, “Fractional repetition codes for repair in distributed storage systems”, 2010
DRESS codes(*)
(Distributed Replication based Exact Simple Storage)

• consist of the concatenation of
  – Outer maximum distance separable (MDS) code
  – And inner fractional repetition (FR) code.
FR code: definition

• An \((n, \alpha, \rho)\) FR code \(C\) is a collection of \(n\) subsets \(N_1, \ldots, N_n\) of \([\theta]\), for \(n\alpha = \rho\theta\), such that
  – \(|N_i| = \alpha\), for \(1 \leq i \leq n\);
  – each element of \([\theta]\) belongs to exactly \(\rho\) subsets in \(C\).
FR code: definition

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  – \(|N_i| = \alpha\), for \(1 \leq i \leq n\);
  – each element of \([\theta]\) belongs to exactly \(\rho\) subsets in \(C\).

• Node \(i\) stores the symbols of MDS codeword indexed by \(N_i\)
[\((\theta, M), k, (n, \alpha, \rho)\)]- DRESS code

Any symbol is replicated \(\rho\) times (repetition degree)

\[
\theta \rho = n\alpha
\]
Any symbol of an MDS codeword is replicated $\rho$ times $c_1, c_2, \ldots, c_\theta$.

Node $1$:
$C_{i_1}, C_{i_2}, \ldots, C_{i_{\alpha}}$

Node $2$:
$C_{j_1}, C_{j_2}, \ldots, C_{j_{\alpha}}$

Node $n$:
$C_{S_1}, C_{S_2}, \ldots, C_{S_{\alpha}}$

To have MDS property:
- The number of symbols of MDS codeword stored on any $k$ nodes $\geq M$.

$\theta \rho = n\alpha$

$\theta, M, k, (n, \alpha, \rho)$ - DRESS code

File $f \in \mathbb{F}_M^\theta$
FR code: MDS property

• To have MDS property:

  # of symbols of MDS codeword stored on any $k$ nodes $\geq M$

  $\iff \left| \bigcup_{i \in I, |I|=k} N_i \right| \geq M$
FR code: MDS property

• To have MDS property:
  
  \[ \text{# of symbols of MDS codeword stored on any } k \text{ nodes } \geq M \]
  
  \[ \iff \left| \bigcup_{i \in I, |I|=k} N_i \right| \geq M \]

• The same FR code can be used in many DRESS codes, with different \( k \)'s
FR code: MDS property

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• For a given \( k \), denote the file size \( M(k) \)
FR code: MDS property

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• The same FR code can be used in many DRESS codes, with different \( k \)’s

• For a given \( k \), denote the file size \( M(k) \)

• Want to store more than MBR code:

  \[ M(k) > k\alpha - \binom{k}{2} \]
DRESS code: maximum file size

- Denote by $A(n, k, \alpha, \rho)$ the upper bound on $M(k)$ for an $[(\theta, M(k)), k, (n, \alpha, \rho)]$-DRESS code

\[
A(n, k, \alpha, \rho) \leq \varphi(k), \text{ where } \\
\varphi(1) = \alpha, \varphi(k + 1) = \varphi(k) + \alpha - \left[\frac{\rho \varphi(k) - k \alpha}{n - k}\right]
\]

(*) Salim El Rouayheb and Kannan Ramchandran, “Fractional repetition codes for repair in distributed storage systems”, 2010
DRESS code: maximum file size

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Is it possible to achieve this bound?

(*) Salim El Rouayheb and Kannan Ramchandran, “Fractional repetition codes for repair in distributed storage systems”, 2010
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\]
\[
\varphi(1) = \alpha, \quad \varphi(k + 1) = \varphi(k) + \alpha - \left\lceil \frac{\rho \varphi(k) - k \alpha}{n-k} \right\rceil
\]

Is it possible to achieve this bound?

- $A(n, k, \alpha, \rho)$ is determined by the parameters of the inner FR code, for a given $k$.

- **Optimal FR** code stores maximum possible file

(*) Salim El Rouayheb and Kannan Ramchandran, “Fractional repetition codes for repair in distributed storage systems”, 2010
Known constructions of FR codes

Our results: optimal FR codes

• $\rho = 2$:
  – Based on complete $r$-partite graphs (Turàn graphs)
  – Based on regular graphs with a given girth (e.g. cage graphs)

• $\rho > 2$:
  – Based on transversal designs
  – Based on biregular bipartite graphs with a given girth (e.g. generalized polygons)
Our results: optimal FR codes

• \( \rho = 2 \):
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Optimal FR codes with $\rho = 2$:

Example: Bipartite graph based FR code

$\mathbf{f} \in \mathbb{F}^M$

$(9, M)$ MDS

$(c_1, c_2, \ldots, c_9)$

$(6, 3, 2) \leftarrow \text{FR}$
Optimal FR codes with $\rho = 2$:

Example: Bipartite graph based FR code

<table>
<thead>
<tr>
<th>$k$</th>
<th>$M(k)$</th>
<th>MBR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>6</td>
</tr>
</tbody>
</table>

$f \in \mathbb{F}_M^k$

$(9, M)$ MDS

$(c_1, c_2, \ldots, c_9)$

$(6, 3, 2) – FR$

A

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

B

$\begin{bmatrix} c_4 \\ c_5 \\ c_6 \end{bmatrix}$

C

$\begin{bmatrix} c_7 \\ c_8 \\ c_9 \end{bmatrix}$

D

$\begin{bmatrix} c_1 \\ c_4 \\ c_7 \end{bmatrix}$

E

$\begin{bmatrix} c_2 \\ c_5 \\ c_8 \end{bmatrix}$

F

$\begin{bmatrix} c_3 \\ c_6 \\ c_9 \end{bmatrix}$
Optimal FR codes with $\rho = 2$:

• $\alpha$ –regular graph $G = (V, E) \iff$ FR code $C_G$ with $\rho = 2$

| $V$ | $n$ = number of nodes |
| $E$ | $\theta$ = length of MDS code |

Degree $\alpha = \text{storage } \alpha \text{ in a node}$
Optimal FR codes with $\rho = 2$:

**Lemma 1**

The file size $M(k)$ of $C_G$ based on $G = (V, E)$ is given by

$$M(k) = k\alpha - \max_{G'=(V',E')\in G_k} |E'|,$$

where $G_k$ is the family of induced subgraphs of $G$ with $k$ vertices.
Lemma 1

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Corollary 1

A graph $G$ contains a $k$-clique iff $M(k) = k\alpha - \binom{k}{2}$ for $C_G$. 

Optimal FR codes with $\rho = 2$: 

$M_k = k\alpha - \max_{G'=(V',E') \in G_k} |E'|$, 

MBR
Optimal FR codes with $\rho = 2$:

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**Corollary 1**
A graph $G$ contains a $k$-clique iff $M(k) = k\alpha - \binom{k}{2}$ for $C_G$.

**Corollary 2**
FR code $C_G$ stores more than MBR code iff $G$ does not contain a $k$-clique.
(\(n, r\))-Turàn graph \(T\) is a complete \(r\)-partite graph, \(r | n\):

- Does not contain a \((r+1)\)-clique
- Regular graph of degree

\[
\alpha = (r - 1) \frac{n}{r}
\]
Turàn graph based FR code

• \((n, r)\)-Turàn graph \(T\) is a complete \(r\)-partite graph, \(r | n\):
  – Does not contain a \((r+1)\)-clique
  – Regular graph of degree
    \[
    \alpha = (r - 1) \frac{n}{r}
    \]
• The file size of the FR code \(C_T\):
  
  \[
  M_{C_T}(k) = k\alpha - \binom{k}{2} + r \binom{b}{2} + bt
  \]

  where \(b = \left\lfloor \frac{k}{r} \right\rfloor\), \(t \equiv k \pmod{r}\)

  by Lemma 1: \(M(k) = k\alpha - \max_{G'=(V',E') \in G_k} |E'|\)
Turàn graph based FR code

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Turán graph based FR code

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Attains the upper bound on the file size for all \(1 \leq k \leq \alpha\)
Optimal FR codes with $\rho = 2$:
FR code based on a graph with large girth

**Lemma 2**
The file size $M(k)$ of $C$ for any $1 \leq k \leq \alpha$ satisfies

$$M(k) \leq k\alpha - (k - 1)$$

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- **Girth** $g$ of a graph $G$ is the length of the shortest cycle
- The file size of $C_G$ is $M(k) = k\alpha - (k - 1)$ iff the girth of $G$ is at least $k + 1$. 
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- **Girth** $g$ of a graph $G$ is the length of the shortest cycle.
- The file size of $C_G$ is $M(k) = k\alpha - (k - 1)$ iff the girth of $G$ is at least $k + 1$.
- The FR code $C_G$ based on an $\alpha$-regular graph $G$ with girth $g$ is **optimal** for each $k \leq g - 1$. 
FR codes with $\rho = 2$

- To obtain a specific value of $M(k)$,

$$k\alpha - \binom{k}{2} \leq M(k) \leq k\alpha - k + 1$$

we need to exclude certain subgraphs from $G$:

$$M(k) \geq k\alpha - \binom{k}{2} + 1 \text{ if } G \text{ does not contain a } k\text{-clique } K_k$$

$$M(k) \geq k\alpha - \binom{k}{2} + 2 \text{ if } G \text{ does not contain a } K_k - e$$
FR codes with $\rho = 2$

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$$M(k) \geq k\alpha - \binom{k}{2} + 2 \text{ if } G \text{ does not contain a } K_{k - 1}$$

• **Turàn type problem:** find the minimum number of vertices in a graph which does not contain a specific subgraph.

  • **Turàn graphs** are the graphs which do not contain a clique and have minimum number of vertices

  • **Cages** are the graphs with the given degree and girth have minimum number of vertices
Bound on reconstruction degree $k$

• Given $M, \theta, n,$ and $\alpha$, find the smallest reconstruction degree $k$ to provide the maximum failure resilience.

• **Lemma.** Let $C$ be an $(n, \alpha, \rho)$ FR code which stores a file of a given size $M$. The reconstruction degree $k$ should satisfy

$$k \geq \left\lfloor \frac{n \left( \begin{array}{c} M - 1 \\ \alpha \end{array} \right)}{\left( \begin{array}{c} \theta \\ \alpha \end{array} \right)} \right\rfloor + 1$$
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\[
k \geq \left\lfloor \frac{n \binom{M-1}{\alpha}}{\binom{\theta}{\alpha}} \right\rfloor + 1
\]

• FR code based on $K_n$ attains this bound for $k = n - 2$

• FR code based on $K_n$ without one perfect matching attains this bound for $k = n - 2$
Optimal FR codes with $\rho > 2$

Transversal Designs based codes
Optimal FR codes with $\rho > 2$

Transversal Designs based codes

- FR codes based on Transversal Designs
  - Nodes = points of the design
  - MDS symbols = blocks of the design
Optimal FR codes with $\rho > 2$

Transversal Designs based codes

• FR codes based on Transversal Designs
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• $(n, \alpha, \rho)$ FR code:
  • There are $n$ points in the design
  • Each block contains $\rho$ points
  • Each point is contained in $\alpha$ blocks
Optimal FR codes with $\rho > 2$

Transversal Designs based codes

- FR codes based on Transversal Designs
  - Nodes = points of the design
  - MDS symbols = blocks of the design
- The file size of the FR code $C_{TD}$ based on TD is strictly larger than for MBR code
- For the parameter $\alpha$ large enough it is an optimal FR code
Allowing parallel reads using our FR codes

- Parallel reads of a subset of any data symbols:
  - Multiple concurrent data collectors
  - Reading at most one element from each node
Allowing parallel reads using our FR codes

- **Parallel reads of a subset of any data symbols:**
  - Multiple concurrent data collectors
  - Reading **at most one** element from each node

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Allowing parallel reads using our FR codes

- **Parallel** reads of a **subset** of any data symbols:
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\{1,2,3,4\} cannot be read in parallel!
Allowing parallel reads using our FR codes

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Allowing parallel reads using our FR codes

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- **Batch codes**(* *) are designed to have this property

Allowing parallel reads using our FR codes

• Parallel reads of a subset of any data symbols:
  – Multiple concurrent data collectors
  – Reading **at most one** element from each node

• **Batch codes** are designed to have this property

• FR codes based on **transversal designs** are good batch codes\(^(*)\):
  – FR code based on $\text{TD}(q - 1, q)$, for a prime power $q$, allows $n - 1 = q^2 - q - 1$ parallel reads

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N. Silberstein and A. Gàl, “Optimal combinatorial batch codes based on block designs”, 2013
Allowing parallel reads using our FR codes

• Parallel reads of a subset of any data symbols:
  – Multiple concurrent data collectors
  – Reading at most one element from each node
• Batch codes are designed to have this property
• FR codes based on transversal designs are good batch codes:
  – FR code based on \( \text{TD}(q - 1, q) \), for a prime power \( q \), allows \( n - 1 = q^2 - q - 1 \) parallel reads

Define FR Batch codes that have the properties of FR and batch codes simultaneously
FR batch code based on TD(3,4)

\[ \alpha = 4, \rho = 3, n = 12, k = 4, M = 11 \]

Any 11 symbols can be read in parallel!
Conclusion

• New constructions of optimal FR codes based on
  – Turàn regular graphs
  – Graphs with large girth
  – Transversal designs

• FR batch codes
  – Multiple parallel reads
Thank you!
Optimal FR codes with $\rho > 2$

**Transversal Designs based codes**

A *transversal design* $\text{TD}(\rho, \alpha)$ of block size $\rho$ and group size $\alpha$ is a triple $(\mathcal{P}, \mathcal{G}, \mathcal{B})$ where

- $\mathcal{P}$ is a set of $\rho \alpha$ points;
- $\mathcal{G}$ is a partition of $\mathcal{P}$ into $\rho$ sets (groups) of size $\alpha$ each;
- $\mathcal{B}$ is a collection of $\rho$-subsets of $\mathcal{P}$ (blocks);
- each block meets each group in exactly one point;
- any pair of points from different groups is contained in exactly one block.

- FR codes based on Transversal Designs
  - Nodes = points of the design ($\rho \alpha$)
  - MDS symbols = blocks of the design ($\alpha^2$)
**Optimal FR codes with $\rho > 2$**

Transversal Designs based codes

- **TD(3,4):** $n=12$ points, $\theta = 16$ blocks of size $\rho = 3$, each point in $\alpha = 4$ blocks
Optimal FR codes with $\rho > 2$

Transversal Designs based codes

- TD(3,4): $n=12$ points, $\theta = 16$ blocks of size $\rho = 3$, each point in $\alpha = 4$ blocks

<table>
<thead>
<tr>
<th>$k$</th>
<th>$R_C$</th>
<th>MBR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
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<tr>
<td>2</td>
<td>7</td>
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<td>4</td>
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</tbody>
</table>

Node 1: 1 2 3 4
Node 2: 5 6 7 8
Node 3: 9 10 11 12
Node 4: 13 14 15 16
Node 5: 1 5 9 13
Node 6: 2 6 10 14
Node 7: 3 7 11 15
Node 8: 4 8 12 16
Node 9: 1 6 12 15
Node 10: 2 5 11 16
Node 11: 3 8 10 13
Node 12: 4 7 9 14
Optimal FR codes with $\rho > 2$

Transversal Designs based codes

- **Theorem:** Let $C_{TD}$ be FR code based on $TD(\rho, \alpha)$
  \[
  R_{C_{TD}}(k) \geq k\alpha - \binom{k}{2} + \rho \binom{b}{2} + bt
  \]
  where $k = b\rho + t, t \leq r - 1$

*Proof:* similar to Turàn graphs
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For $\alpha \geq \alpha_0(k, \rho)$:

- $R_{C_{TD}}(k) = k\alpha - \binom{k}{2} + \rho \binom{b}{2} + bt$
- Attains the upper bound on FR capacity