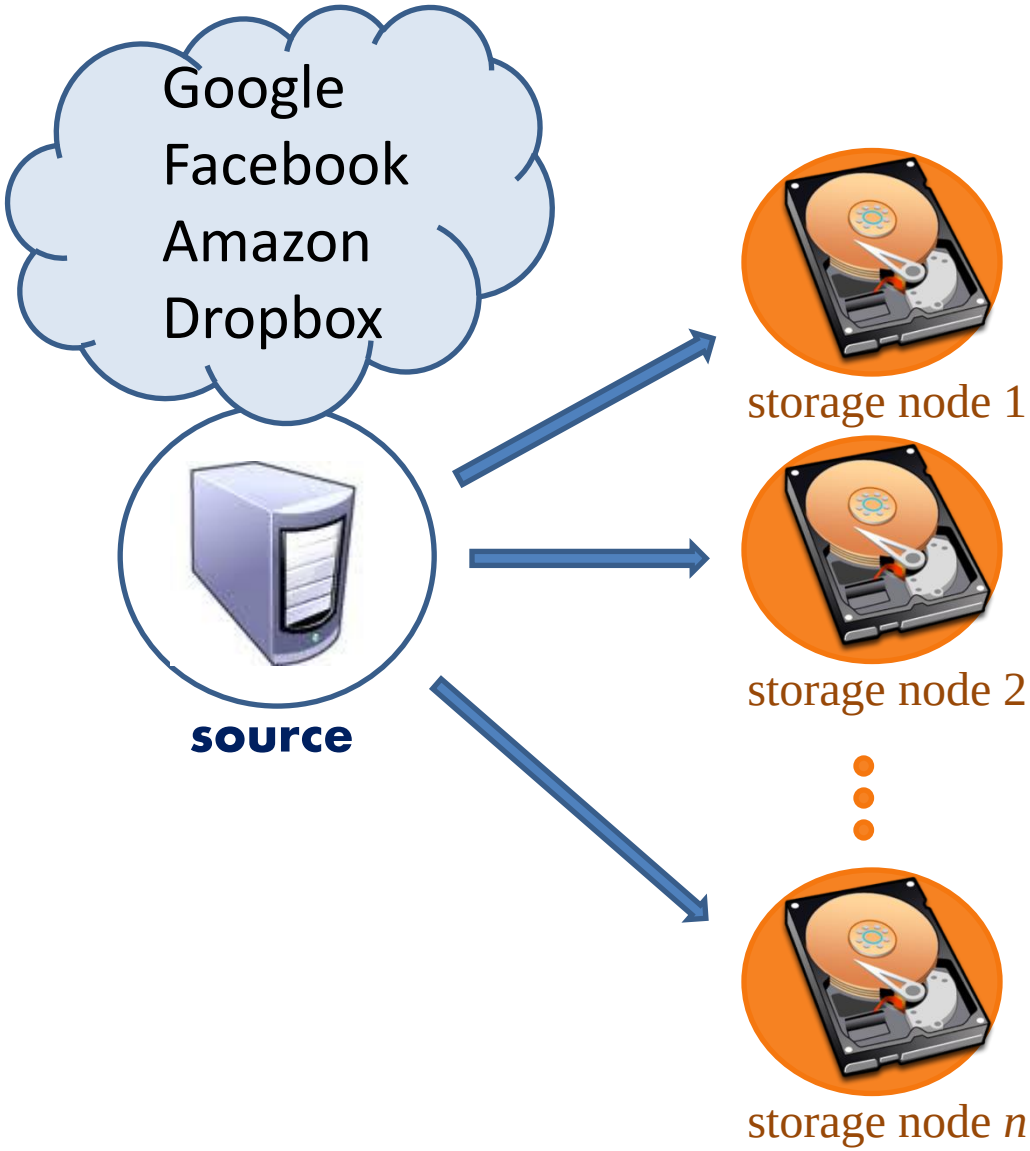
Optimal Fractional Repetition Codes for Distributed Storage Systems

Natalia Silberstein

Joint work with Tuvi Etzion

Distributed storage system (DSS)



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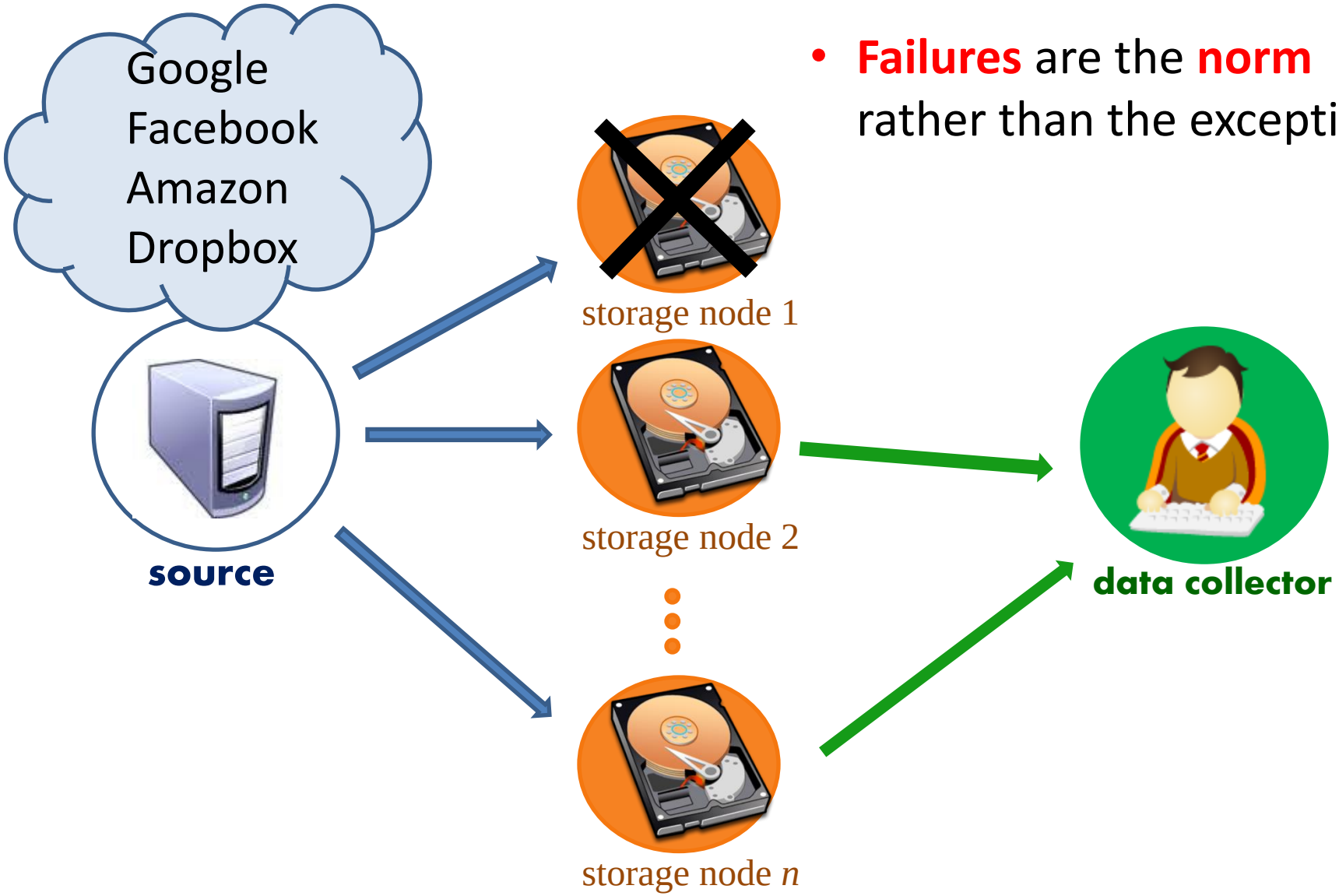

- **Failures** are the **norm** rather than the exception

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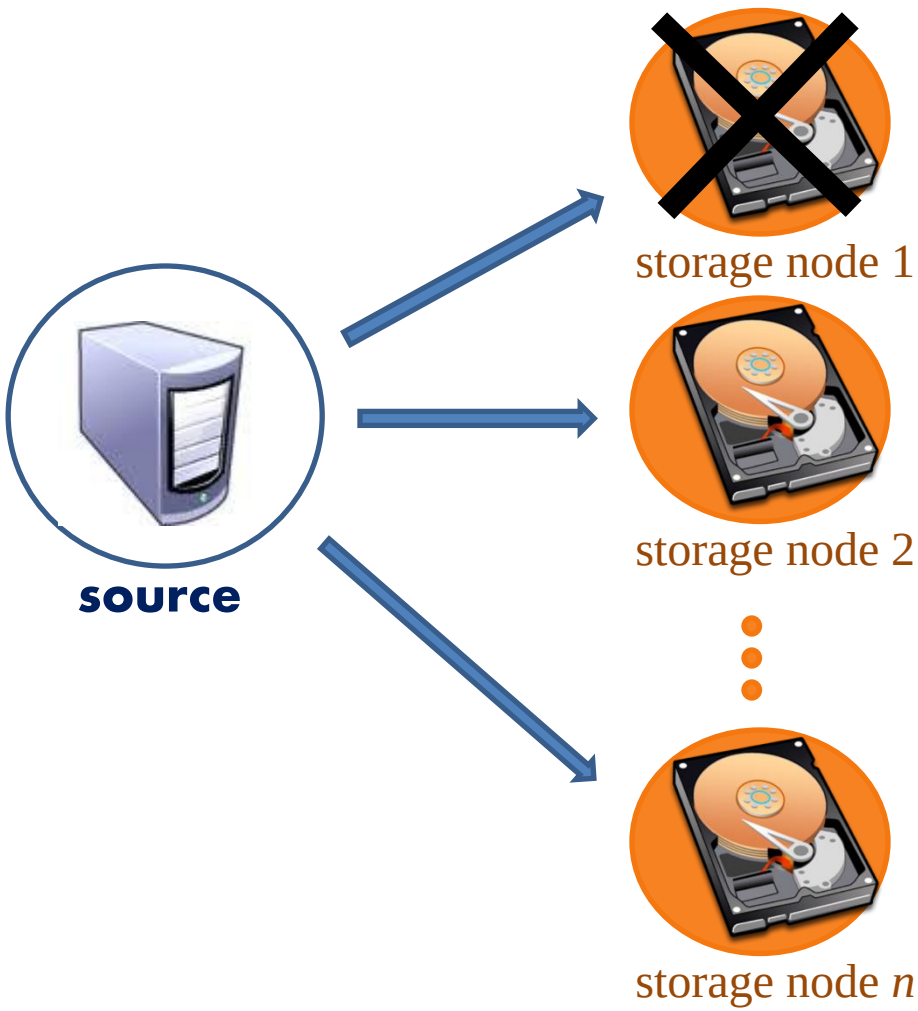
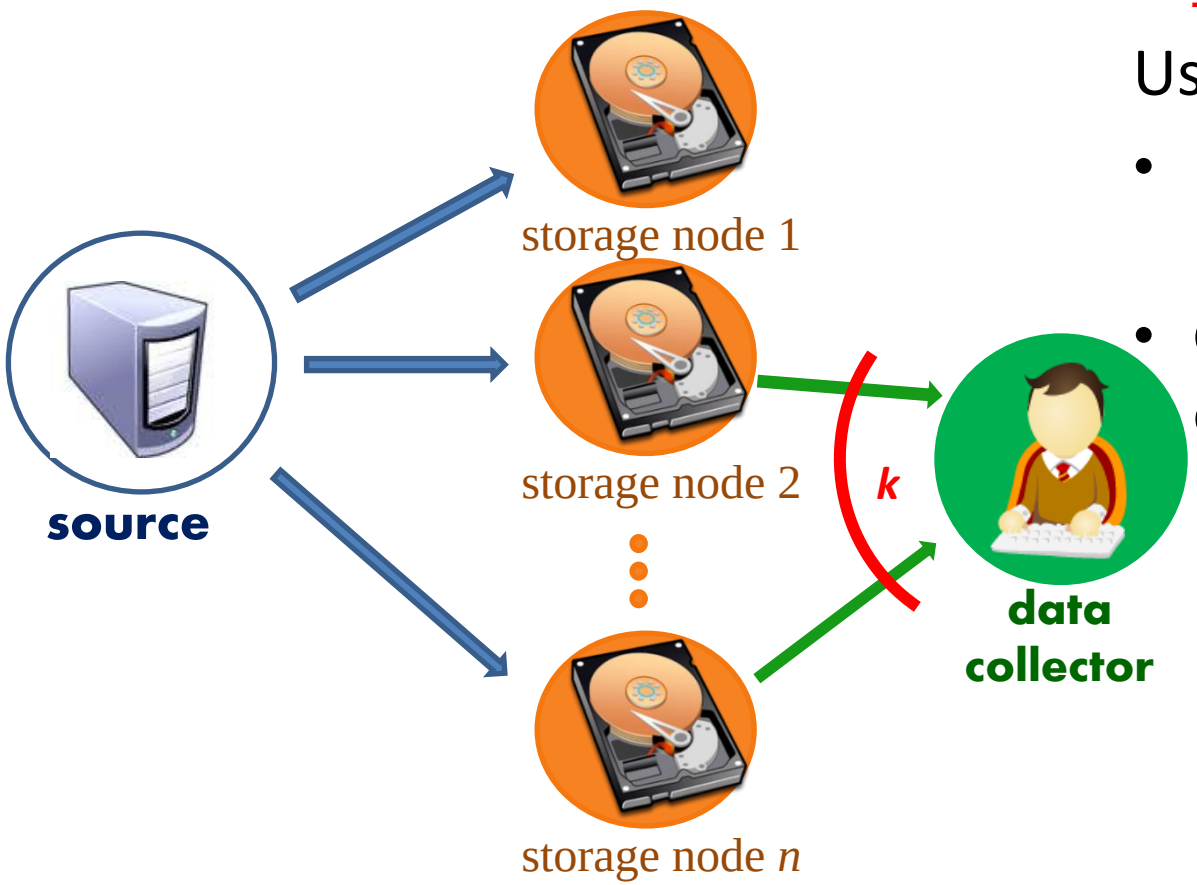


- **Failures** are the **norm** rather than the exception
- **Redundancy** for reliability
  - **Replication**
  - **Erasure coding**

Coding for distributed storage system



- **Erasure codes:**

Using (n, k) erasure code:

- Partition the data into k blocks
- Generate n blocks, store each block in a new node

(n, k) MDS property:

Reconstruct the data from any k nodes

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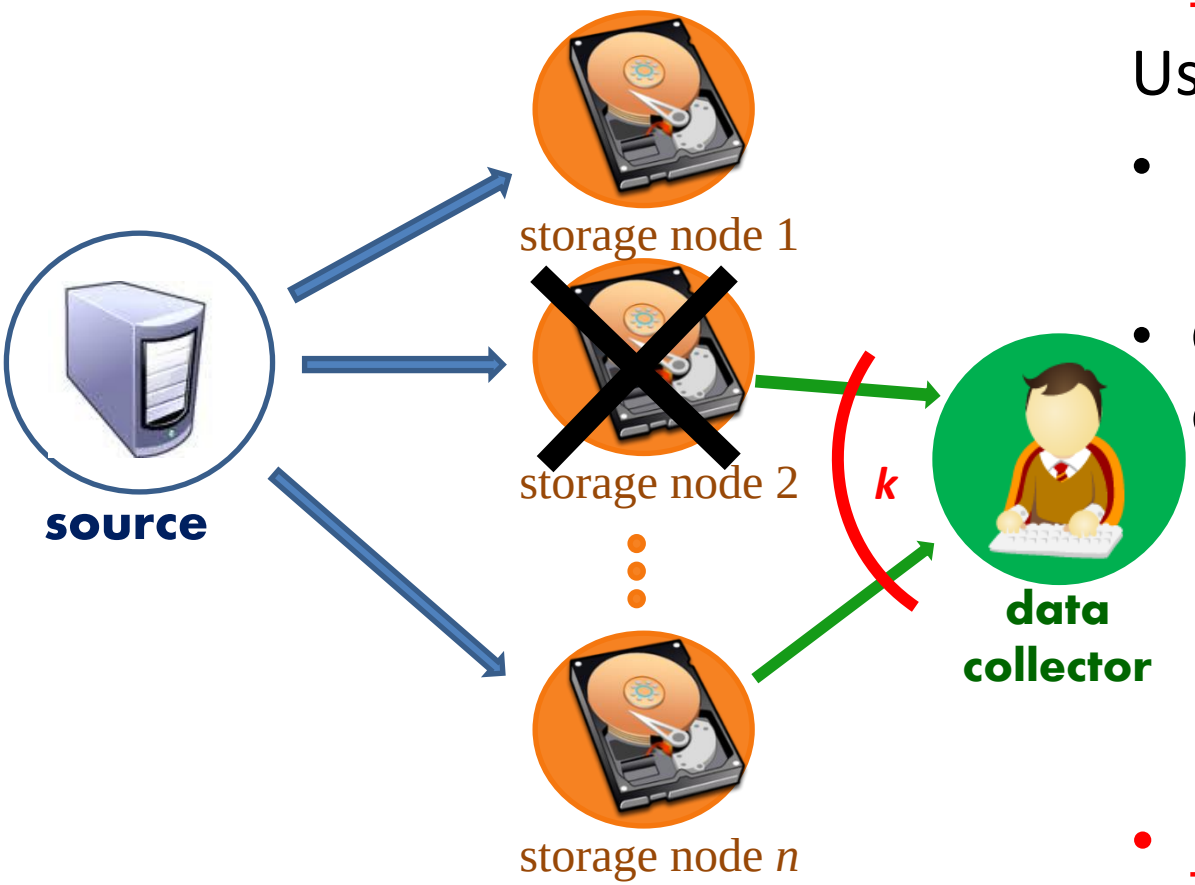

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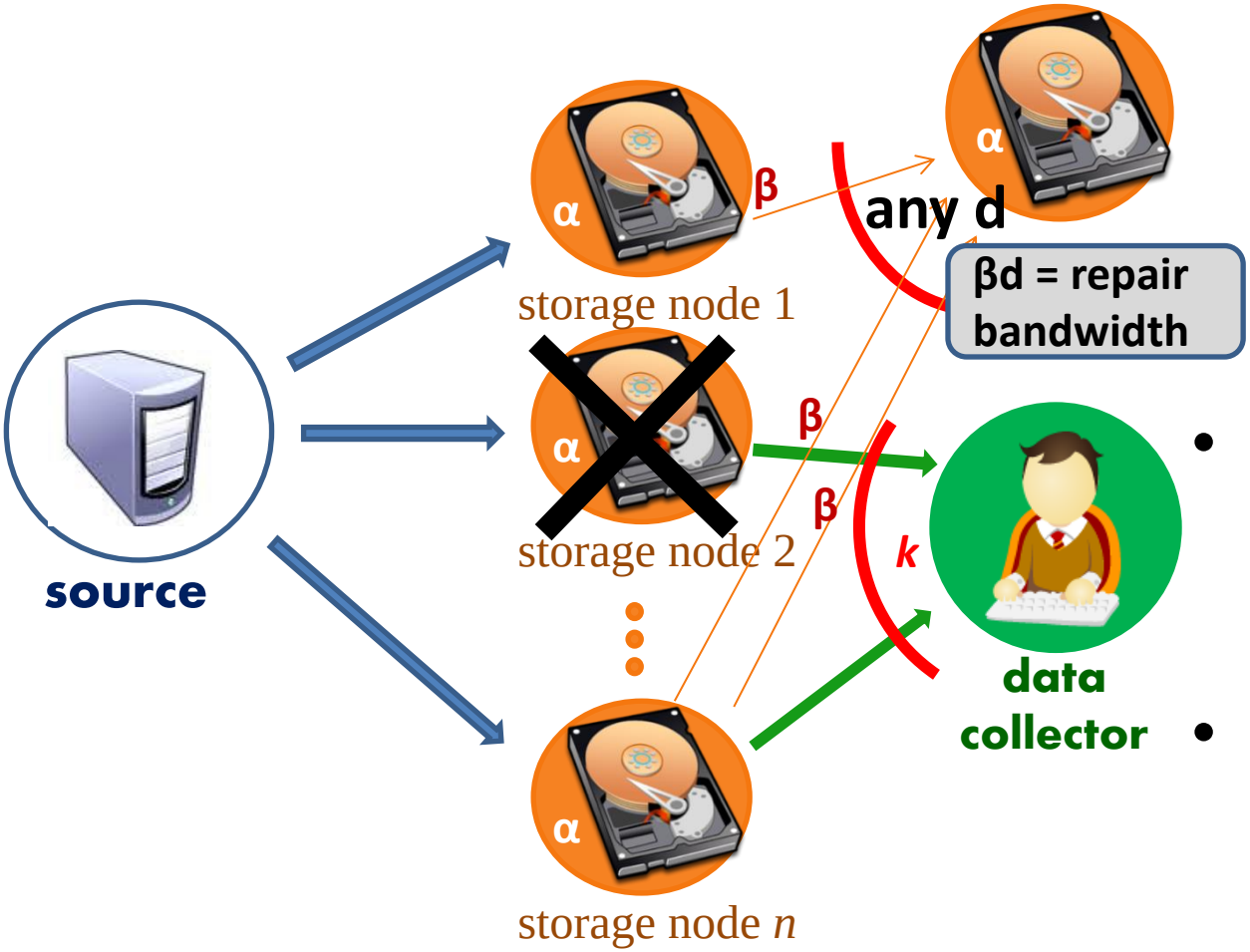
- **Regenerating codes**

efficient node repairs

Regenerating codes (*)



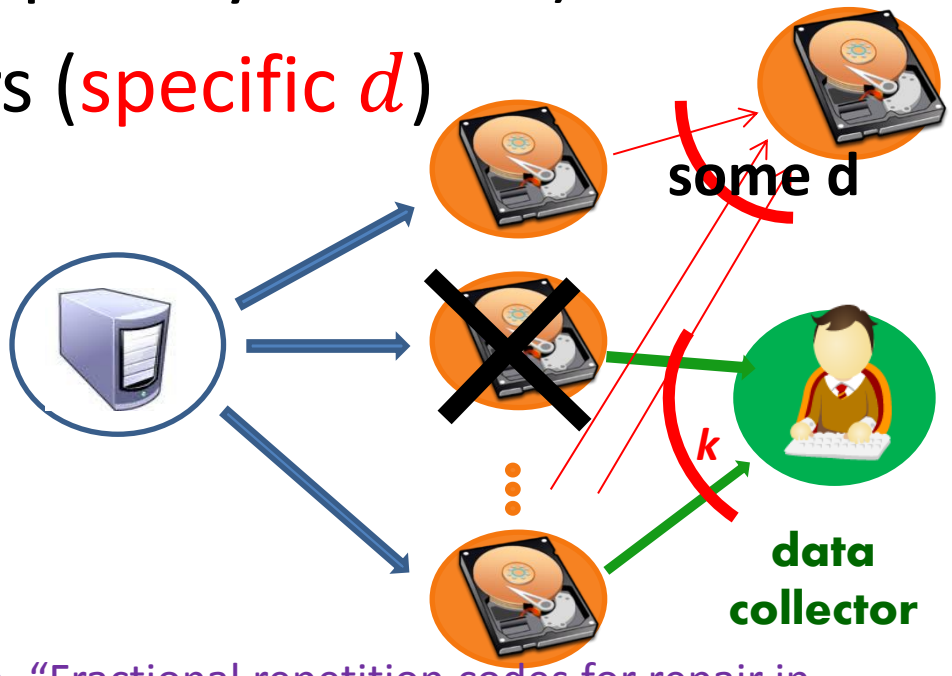
- Minimum storage regenerating (MSR) codes
- Minimum bandwidth regenerating (MBR) codes

(*) A. G. Dimakis, P. B. Godfrey, M. J. Wainwright and K. Ramchandran, "Network coding for distributed storage systems, 2007"

DRESS codes(*)

(Distributed Replication based Exact Simple Storage)

- Minimum repair bandwidth (like MBR)
- Uncoded repair (repair by transfer)
- Table based repairs (specific d)



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Allow to store more data than MBR codes!

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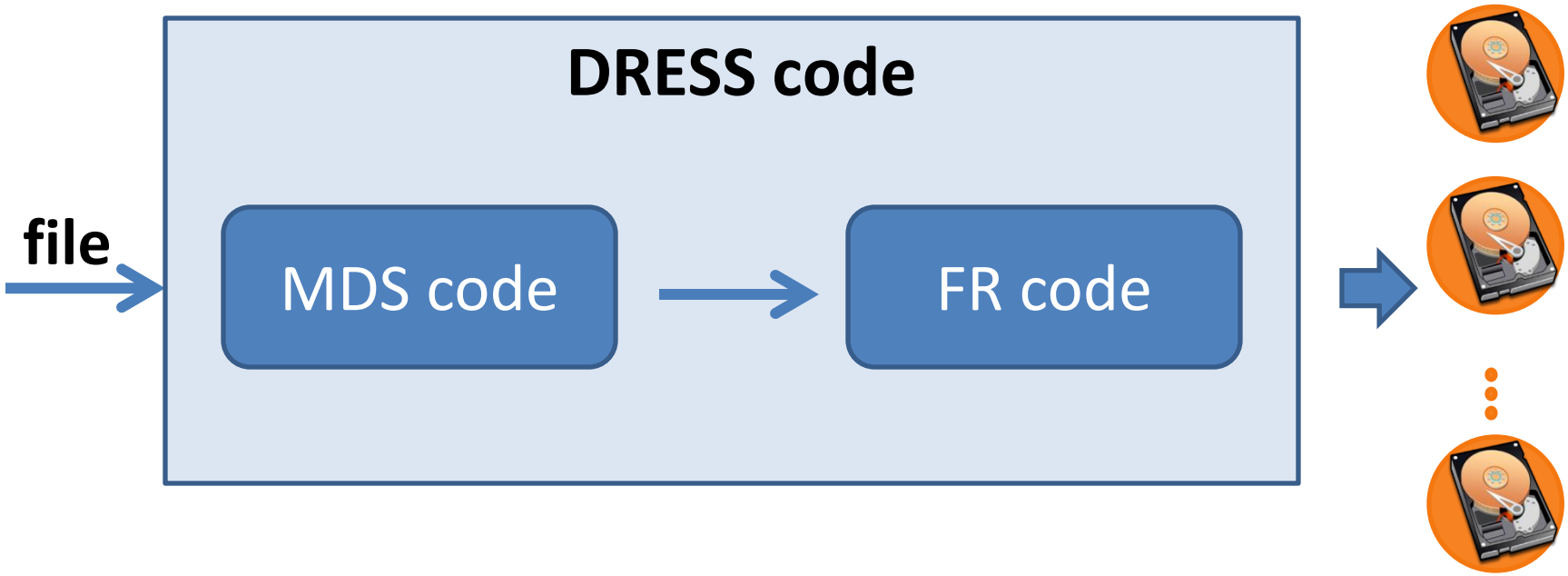
(Distributed Replication based Exact Simple Storage)

- consist of the concatenation of
  - **Outer** maximum distance separable (**MDS**) code
  - And **inner** fractional repetition (**FR**) code.



FR code: definition

- An (n, α, ρ) FR code C is a collection of n subsets $N_1, \ldots, N_n$ of $[\theta]$, for $n\alpha = \rho\theta$, such that
  - $|N_i| = \alpha$, for $1 \leq i \leq n$;
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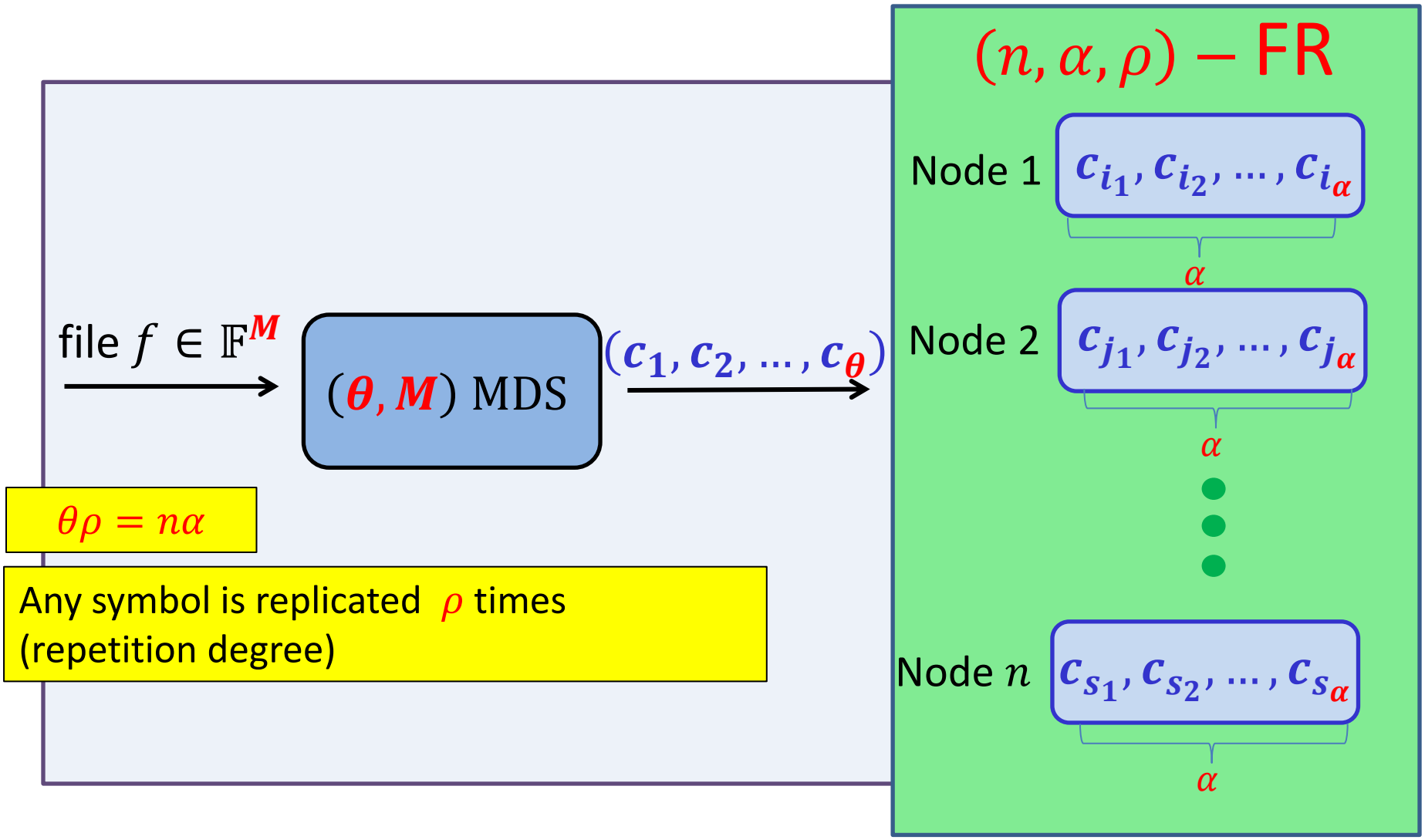
- Node i stores the symbols of MDS codeword indexed by N_i

$[(\theta, M), k, (n, \alpha, \rho)]$- DRESS code



$[(\theta, M), k, (n, \alpha, \rho)]$- DRESS code

file $f \in \mathbb{F}^{M}$ → (θ, M) MDS → $(c_1, c_2, \ldots, c_\theta)$

(n, α, ρ) – FR

Node 1 $c_{i_1}, c_{i_2}, \ldots, c_{i_\alpha}$ (α)

Node 2 $c_{j_1}, c_{j_2}, \ldots, c_{j_\alpha}$ (α)

MDS property (k)

Node n $c_{s_1}, c_{s_2}, \ldots, c_{s_\alpha}$ (α)

$\theta\rho = n\alpha$

Any symbol is replicated ρ times (repetition degree)

To have MDS property:

# of symbols of MDS codeword stored on any *k* nodes $\geq M$

FR code: MDS property

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  $$\Leftrightarrow \left|\cup_{i\in I,|I|=k} N_i\right| \geq M$$

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- The same FR code can be used in many DRESS codes, with different k's
- For a given k, denote the file size $M(k)$
- Want to store more than MBR code:

$$M(k) > k\alpha - \binom{k}{2}$$

MBR

DRESS code: maximum file size

- Denote by $A(n,k,\alpha,\rho)$ the upper bound on $M(k)$ for an $[(\theta, M(k)), k, (n,\alpha,\rho)]$- DRESS code

$$\boldsymbol{A(n,k,\alpha,\rho) \leq \varphi(k),} \text{ where}$$
$$\varphi(1) = \alpha, \varphi(k+1) = \varphi(k) + \alpha - \left\lceil \frac{\rho\varphi(k) - k\alpha}{n-k} \right\rceil^{(*)}$$

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Is it possible to achieve this bound?

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Is it possible to achieve this bound?

- $A(n,k,\alpha,\rho)$ is determined by the parameters of the inner FR code, for a given k.
- **Optimal FR** code stores maximum possible file

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Known constructions of FR codes

- S. El Rouayheb and K. Ramchandran, *"Fractional repetition codes for repair in distributed storage systems,"* Allerton 2010.
- J. C. Koo and J. T. Gill. III, *"Scalable constructions of fractional repetition codes in distributed storage systems,"* Allerton 2011.
- O. Olmez and A. Ramamoorthy, *"Repairable replication-based storage systems using resolvable designs,"* Allerton 2012.
- S. Anil, M. K. Gupta, and T. A. Gulliver, *"Enumerating some fractional repetition codes,"* arXiv 2013.
- S. Pawar, N. Noorshams, S. El Rouayheb, and K. Ramchandran, *"Dress codes for the storage cloud: Simple randomized constructions,"* ISIT 2013.
- B. Zhu, K. W. Shum, H. Li, and H. Hou, "General fractional repetition codes for distributed storage systems" IEEE Communications Letters, Apr. 2014.

Our results: optimal FR codes

- $\boldsymbol{\rho = 2}$**:**
  - Based on complete *r*-partite graphs (Turàn graphs)
  - Based on regular graphs with a given girth (e.g. cage graphs)

- $\boldsymbol{\rho > 2}$**:**
  - Based on transversal designs
  - Based on biregular bipartite graphs with a given girth (e.g. generalized polygons)

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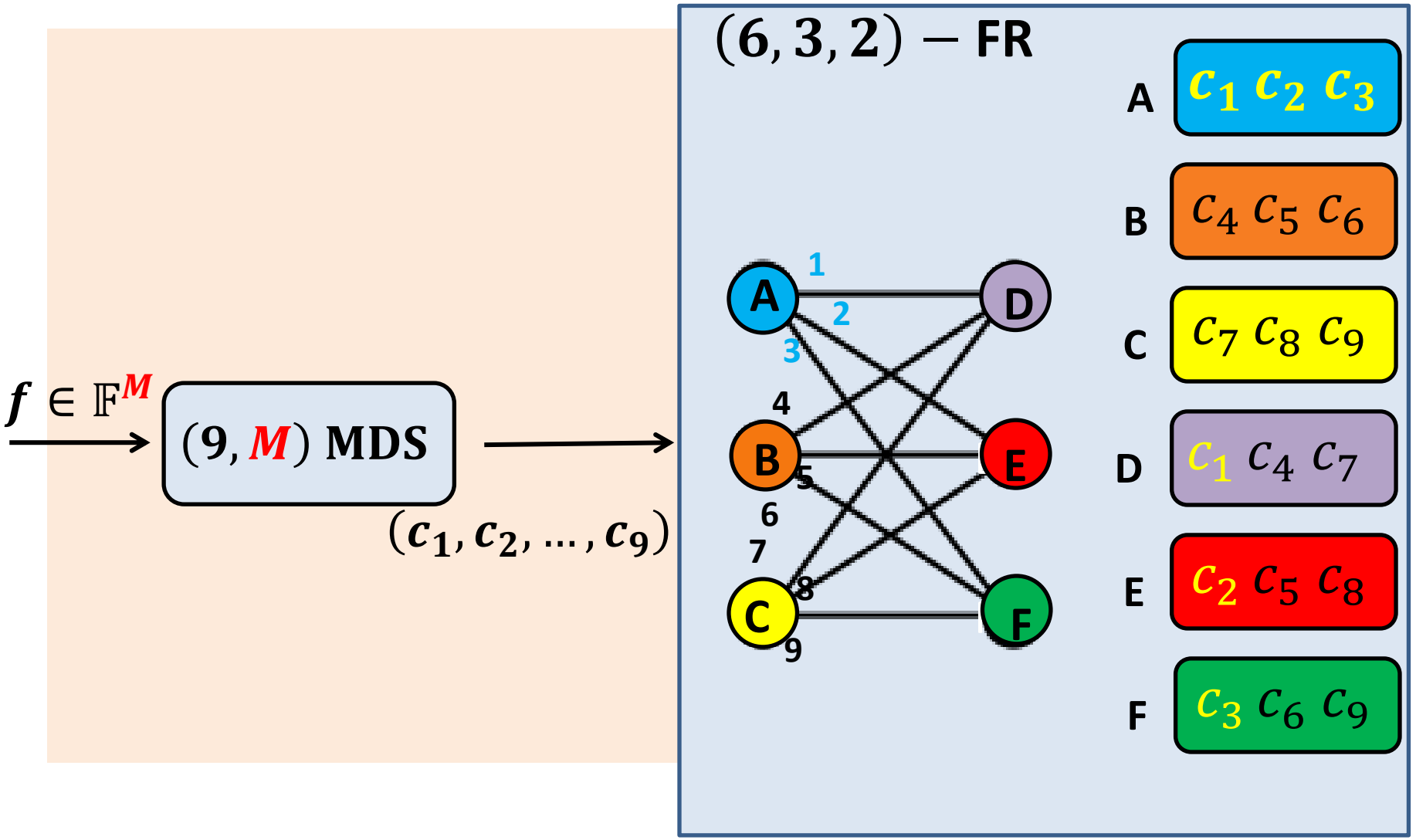
Optimal FR codes with $\rho = 2$:

Example: Bipartite graph based FR code

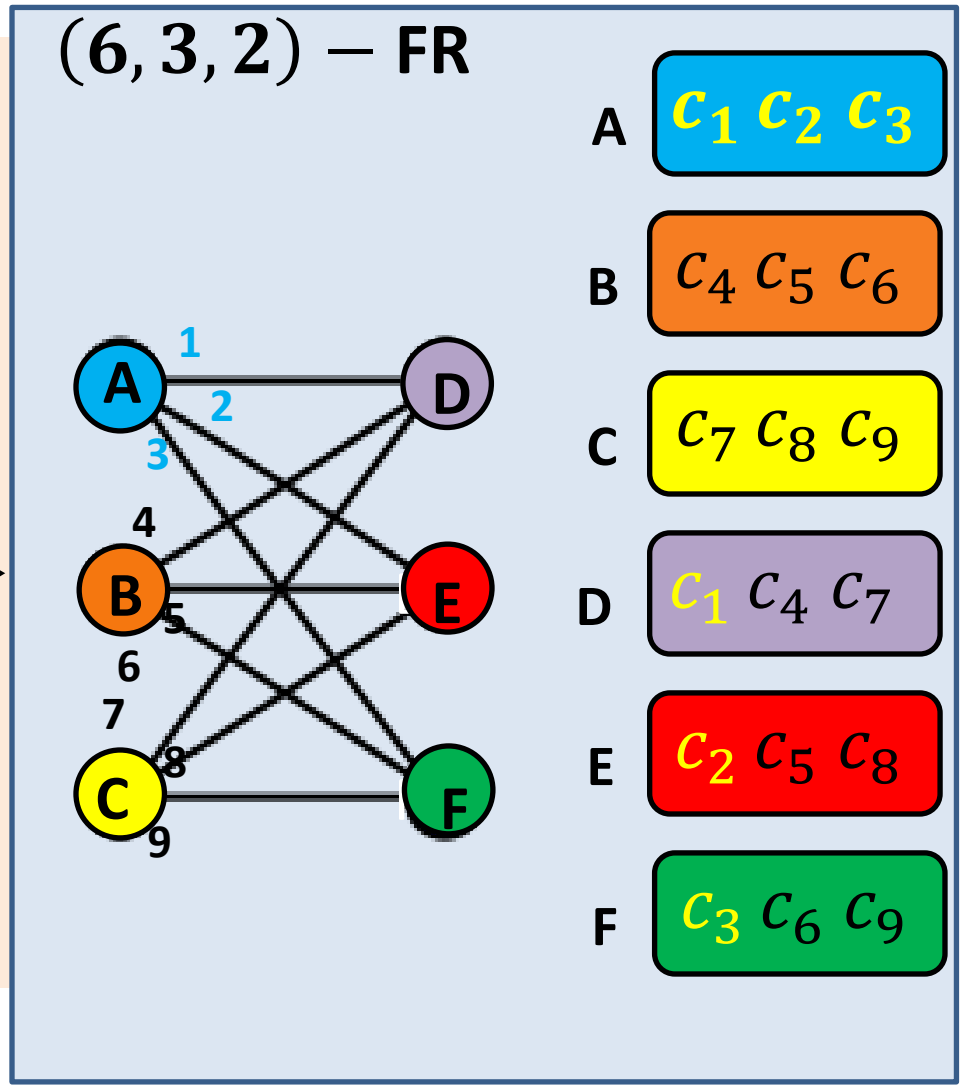


Optimal FR codes with $\rho = 2$:

Example: Bipartite graph based FR code

k	$M(k)$	MBR
1	3	3
2	5	5
3	**7**	**6**

$f \in \mathbb{F}^{M}$ → $(9, M)$ MDS → $(c_1, c_2, \ldots, c_9)$



Optimal FR codes with $\rho = 2$:

- α –regular graph $G = (V, E)$ ⇔ FR code C_G with $\rho = 2$

Vertex $v \in V$ ⇔ node N_i

Edge $e \in E$ ⇔ symbol of an MDS code

$|V| = n$ = number of nodes

$|E| = \theta$ = length of MDS code

Degree α = storage α in a node

Optimal FR codes with $\rho = 2$:

Lemma 1

The file size $M(k)$ of C_G based on $G = (V, E)$ is given by

$$\boldsymbol{M(k) = k\alpha - max_{G'=(V',E')\in G_k} |E'|},$$

where G_k is the family of induced subgraphs of G with k vertices.

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A graph G contains a k-clique *iff* $M(k) = k\alpha - \binom{k}{2}$ for C_G.

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MBR

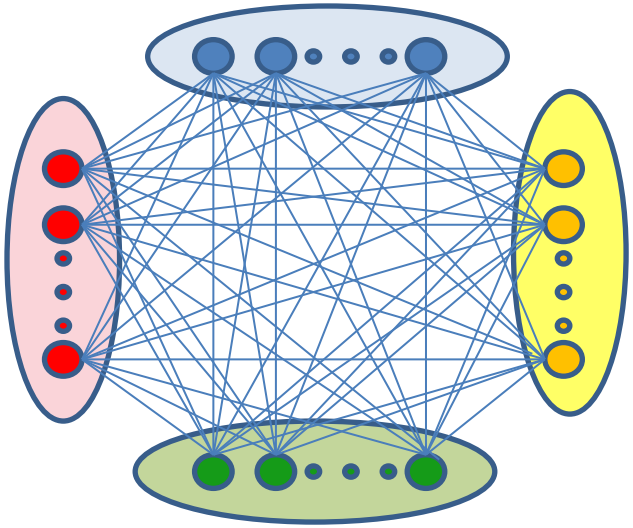
Corollary 2

FR code C_G stores more than MBR code *iff* G does not contain a k-clique.

Turàn graph based FR code

- (n, r)-Turàn graph *T* is a complete r-partite graph, $r|n$:
  - Does not contain a $(r+1)$-clique
  - Regular graph of degree

$$\alpha = (r - 1)\frac{n}{r}$$



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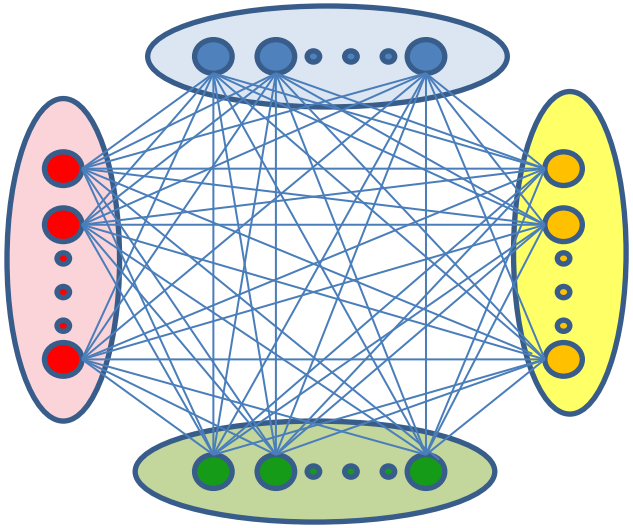

- The file size of the FR code C_T:

$$M_{C_T}(k) = k\alpha - \binom{k}{2} + r\binom{b}{2} + bt$$

where $b = \left\lfloor \frac{k}{r} \right\rfloor, t \equiv k(mod\ r)$

by Lemma 1: $\boldsymbol{M(k) = k\alpha - \max_{G'=(V',E')\in G_k} |E'|}$

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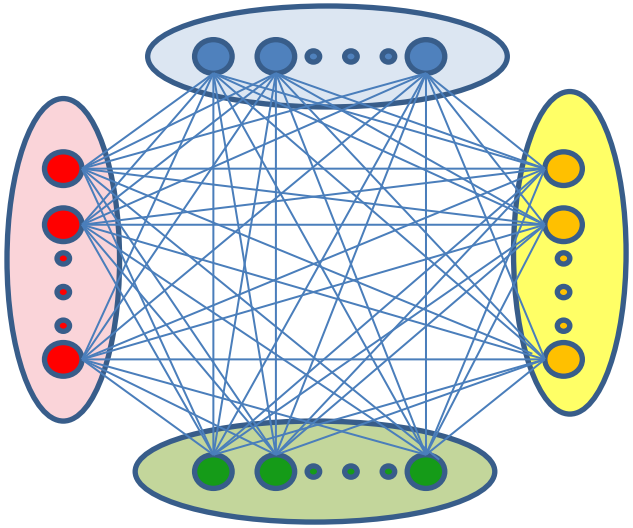

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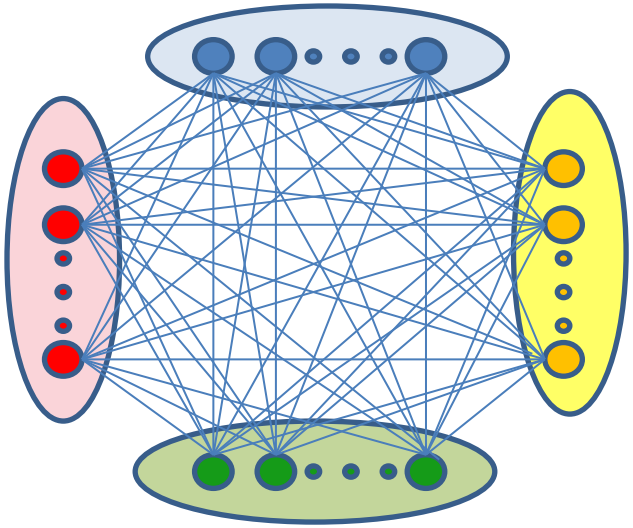

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Attains the upper bound on the file size for all $1 \leq k \leq \alpha$

Optimal FR codes with $\rho = 2$: FR code based on a graph with large girth

Lemma 2

The file size $M(k)$ of C for any $1 \leq k \leq \alpha$ satisfies

$$\boldsymbol{M(k) \leq k\alpha - (k-1)}$$

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- **Girth** g of a graph G is the length of the shortest cycle
- The file size of C_G is $M(k) = k\alpha - (k-1)$ *iff* the girth of G is at least $k+1$.

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- **Girth** g of a graph G is the length of the shortest cycle
- The file size of C_G is $M(k) = k\alpha - (k-1)$ *iff* the girth of G is at least $k+1$.
- The FR code C_G based on an α-regular graph G with girth g is **optimal** for each $k \leq g - 1$.

FR codes with $\rho = 2$

- To obtain a specific value of $M(k)$,

$$k\alpha - \binom{k}{2} \leq M(k) \leq k\alpha - k + 1$$

we need to exclude certain subgraphs from *G*:

$M(k) \geq k\alpha - \binom{k}{2} + 1$ if *G* does not contain a *k*-clique K_k

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- Turàn type problem: find the minimum number of vertices in a graph which does not contain a specific subgraph.
  - Turàn graphs are the graphs which do not contain a clique and have minimum number of vertices
  - Cages are the graphs with the given degree and girh have minimum number of vertices

Bound on reconstruction degree *k*

- Given $M, \theta, n,$ and α, find the smallest reconstruction degree k to provide the maximum failure resilience.
- *Lemma.* Let *C* be an (n, α, ρ) FR code which stores a file of a given size *M*. The reconstruction degree *k* should satisfy

$$k \geq \left\lceil \frac{n\binom{M-1}{\alpha}}{\binom{\theta}{\alpha}} \right\rceil + 1$$

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- FR code based on K_n attains this bound for $k = n - 2$
- FR code based on K_n without one perfect matching attains this bound for $k = n - 2$

Optimal FR codes with $\rho > 2$
Transversal Designs based codes

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- FR codes based on Transversal Designs
  - Nodes = points of the design
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- (n, α, ρ) FR code:
  - There are n points in the design
  - Each block contains ρ points
  - Each point is contained in α blocks

Optimal FR codes with $\rho > 2$ Transversal Designs based codes

- FR codes based on Transversal Designs
  - Nodes = points of the design
  - MDS symbols = blocks of the design
- The file size of the FR code C_{TD} based on TD is strictly larger than for MBR code
- For the parameter α large enough it is an optimal FR code

Allowing parallel reads using our FR codes

- Parallel reads of a subset of any data symbols:
  - Multiple concurrent data collectors
  - Reading **at most one** element from each node

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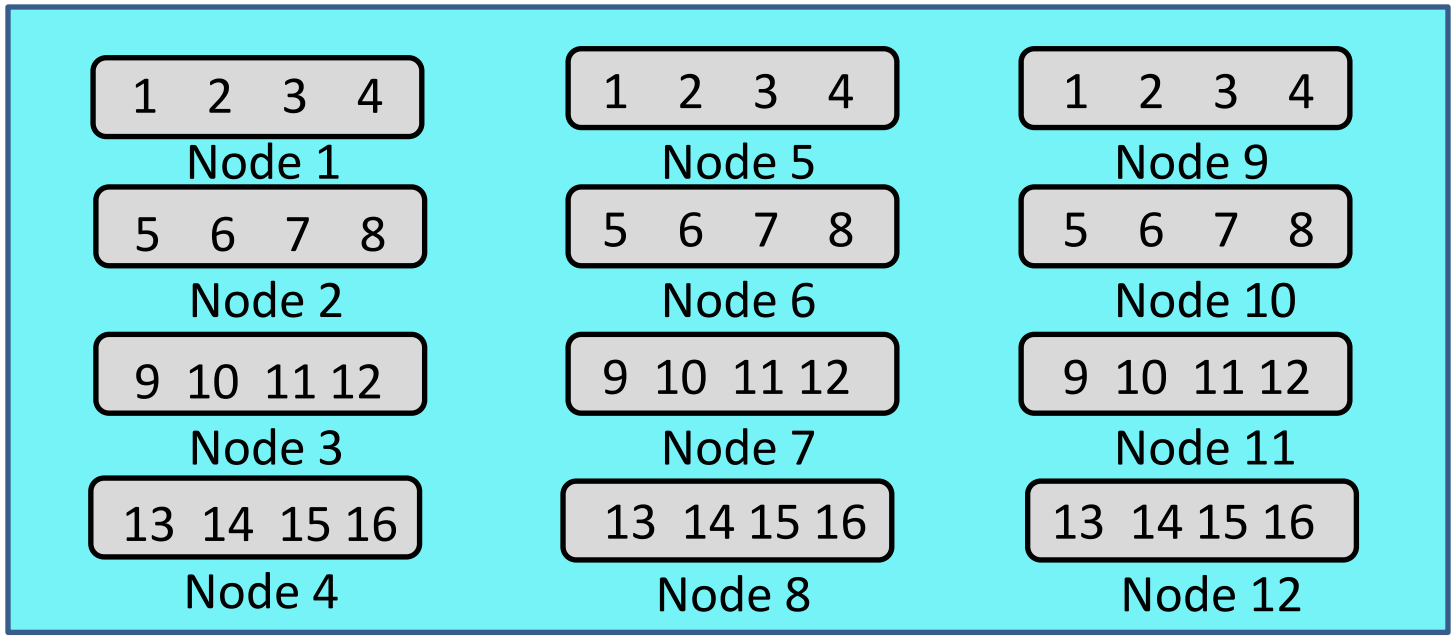

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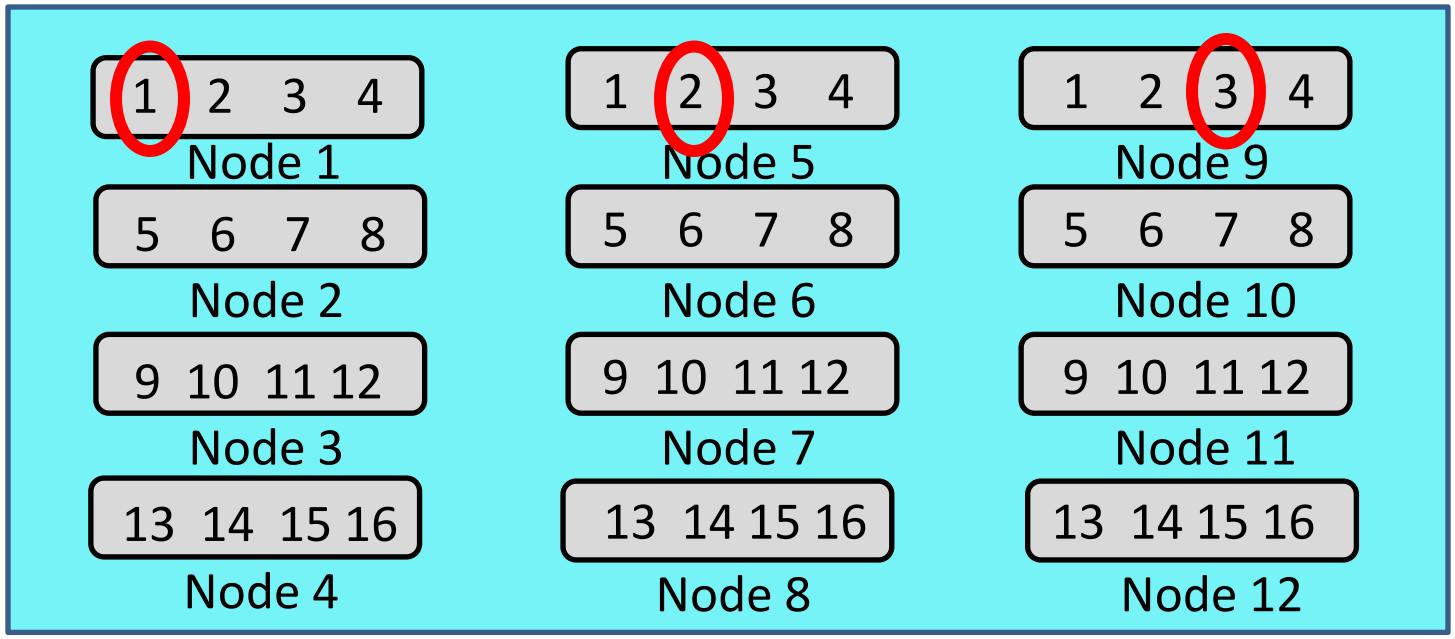

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(*) Y. Ishai, E. Kushilevitz, R. Ostrovsky, and A. Sahai, "Batch codes and their applications", 2004

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- Batch codes are designed to have this property
- FR codes based on transversal designs are good batch codes(*):
  - FR code based on $\mathrm{TD}(q-1, q)$, for a prime power q, allows $n-1 = q^2 - q - 1$ parallel reads

N. Silberstein and A. Gàl, "Optimal combinatorial batch codes based on block designs", 2013

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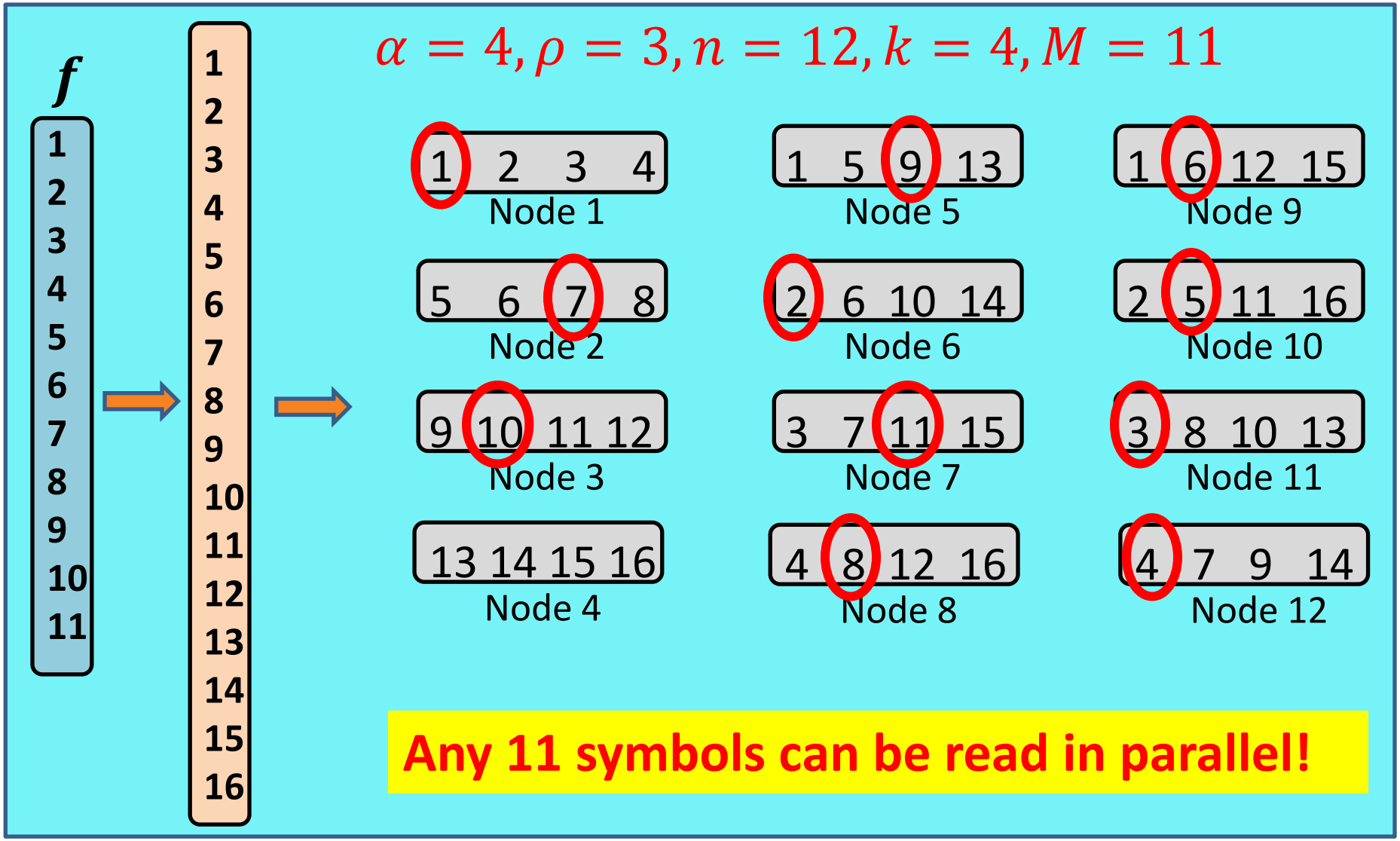
Define **FR Batch** codes that have the properties of **FR** and **batch** codes simultaneously

FR batch code based on TD(3,4)



Conclusion

- New constructions of optimal FR codes based on
  - Turàn regular graphs
  - Graphs with large girth
  - Transversal designs
- FR batch codes
  - Multiple parallel reads

Thank you!

Optimal FR codes with $\rho > 2$ Transversal Designs based codes

A transversal design $\mathrm{TD}(\rho, \alpha)$ of block size ρ and group size α is a triple $(\mathcal{P}, \mathcal{G}, \mathcal{B})$ where

- $\mathcal{P}$ is a set of $\rho\alpha$ points;
- $\mathcal{G}$ is a partition of $\mathcal{P}$ into ρ sets (groups) of size α each;
- $\mathcal{B}$ is a collection of ρ-subsets of $\mathcal{P}$(blocks);
- each block meets each group in exactly one point;
- any pair of points from different groups is contained in exactly one block.

- FR codes based on Transversal Designs
  - Nodes = points of the design ($\rho\alpha$)
  - MDS symbols = blocks of the design (α^2)

Optimal FR codes with $\rho > 2$
Transversal Designs based codes

- TD(3,4): n=12 points, θ =16 blocks of size ρ =3, each point in α =4 blocks

$$\left(\begin{array}{cccc}
1111 & 0000 & 0000 & 0000 \\
0000 & 1111 & 0000 & 0000 \\
0000 & 0000 & 1111 & 0000 \\
0000 & 0000 & 0000 & 1111 \\
\hline
1000 & 1000 & 1000 & 1000 \\
0100 & 0100 & 0100 & 0100 \\
0010 & 0010 & 0010 & 0010 \\
0001 & 0001 & 0001 & 0001 \\
\hline
1000 & 0100 & 0001 & 0010 \\
0100 & 1000 & 0010 & 0001 \\
0010 & 0001 & 0100 & 1000 \\
0001 & 0010 & 1000 & 0100
\end{array}\right)$$

1 2 3 4 Node 1	1 5 9 13 Node 5	1 6 12 15 Node 9
5 6 7 8 Node 2	2 6 10 14 Node 6	2 5 11 16 Node 10
9 10 11 12 Node 3	3 7 11 15 Node 7	3 8 10 13 Node 11
13 14 15 16 Node 4	4 8 12 16 Node 8	4 7 9 14 Node 12

Optimal FR codes with $\rho > 2$
Transversal Designs based codes

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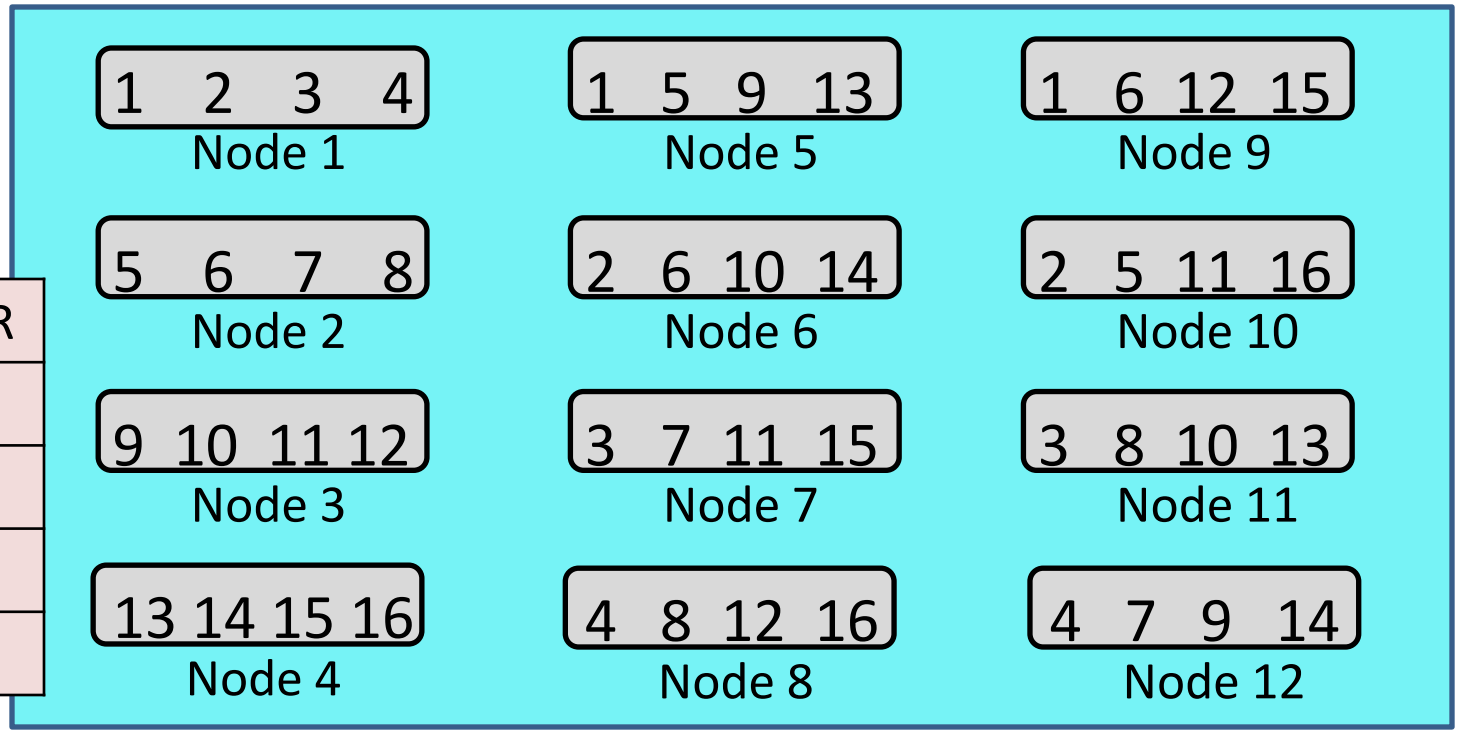
k	R_C	MBR
1	4	4
2	7	7
3	9	9
4	**11**	**10**



Optimal FR codes with $\rho > 2$
Transversal Designs based codes

- Theorem: Let C_{TD} be FR code based on $\mathrm{TD}(\rho, \alpha)$

$$R_{C_{TD}}(k) \geq k\alpha - \binom{k}{2} + \rho\binom{b}{2} + bt$$

where $k = b\rho + t, t \leq r - 1$

Proof: similar to Turàn graphs

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For $\alpha \geq \alpha_0(k, \rho)$:

- $R_{C_{TD}}(k) = k\alpha - \binom{k}{2} + \rho\binom{b}{2} + bt$
- Attains the upper bound on FR capacity