

Channel Upgradation for Non-Binary Input Alphabets and MACs

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Problem

$$\begin{array}{ccc} W : \mathcal{X} \rightarrow \mathcal{Y} & & |\mathcal{Y}| \gg S \\ & \Downarrow \text{upgrade}(W, S) & \\ W' : \mathcal{X} \rightarrow \mathcal{Y}' & & |\mathcal{Y}'| \leq S \end{array}$$

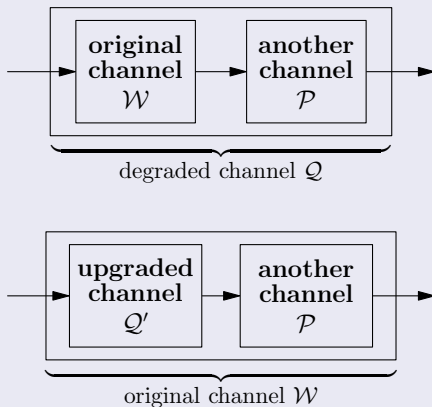
Goal: make $I(W') - I(W)$ as small as possible, where $I(\cdot)$ is symmetric capacity.

- \mathcal{X} need not be binary
- W need not be symmetric
- W can be a t -user MAC $W : \mathcal{X}^t \rightarrow \mathcal{Y}$

Concurrent work

A. Ghayoori and T. A. Gulliver, Upgraded Approximation of Non-Binary Alphabets for Polar Code Construction, on arXiv

Upgradation Vs. Degradation



- $W(y|u)$ is the probability of receiving $y \in \mathcal{Y}$, given that $u \in \mathcal{X}$ was sent.
- The input is assumed to have symmetric distribution over \mathcal{X} .

Motivation: polar codes

- Assure Eve's channels are *very* bad (wiretap setting)
- Upper bound on rate (error correcting code)

Theorem

Let $W : \mathcal{X} \rightarrow \mathcal{Y}$ be a given channel, with $|\mathcal{X}| = q$.

Let $\mu \geq \max\{5, q(q-1)\}$ be a given fidelity parameter.

We can construct $W' : \mathcal{X} \rightarrow \mathcal{Y}'$ such that

- W' is upgraded with respect to W
- $I(W') - I(W) \leq \frac{q-1}{\mu}(2 + q \cdot \ln q)$
- $|\mathcal{Y}'| \leq S(q, \mu) \triangleq (2\mu)^q + q$

Key ideas

- Use bins:
 - “close” output symbols of W go into the same bin
 - “merge” all symbols in bin to one symbol
- Use “boost” symbols (sparingly):
 - New output alphabet will have $q = |\mathcal{X}|$ special symbols, $\{\kappa_u\}_{u \in \mathcal{X}}$
 - Symbol κ_u received on W' only if u sent
 - Small probability of receiving κ_u

W

	y_1	y_2	...
$u=0$	$W(y_1 0)$	$W(y_2 0)$	
$u=1$	$W(y_1 1)$	$W(y_2 1)$	
$u=2$	$W(y_1 2)$	$W(y_2 2)$	

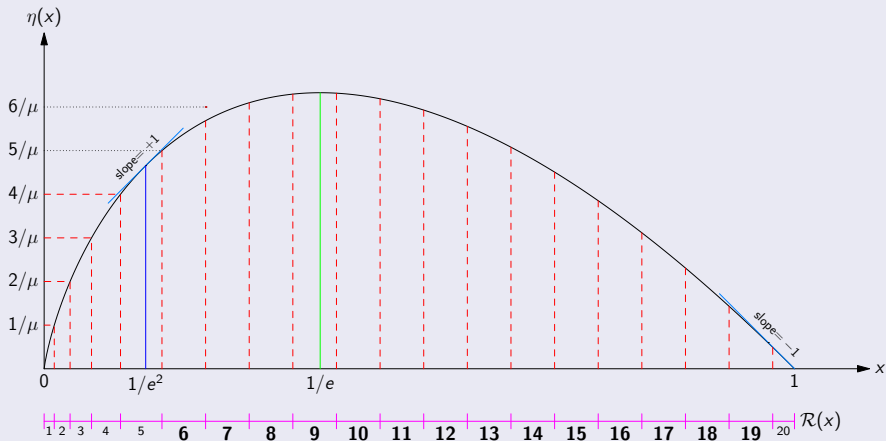
Q

	y_1	y_2	...	κ_0	κ_1	κ_2
$u=0$	$\alpha_0(y_1) \cdot W(y_1 0)$	$\alpha_0(y_2) \cdot W(y_2 0)$		ϵ_0	0	0
$u=1$	$\alpha_1(y_1) \cdot W(y_1 1)$	$\alpha_1(y_2) \cdot W(y_2 1)$		0	ϵ_1	0
$u=2$	$\alpha_2(y_1) \cdot W(y_1 2)$	$\alpha_2(y_2) \cdot W(y_2 2)$		0	0	ϵ_2

$$\epsilon_u = \sum_{y \in \mathcal{Y}} (1 - \alpha_u(y)) W(y|u),$$

$$0 \leq \alpha_u(y) \leq 1, \quad \alpha_u(y) \approx 1$$

Functions $\eta(x)$ and $\mathcal{R}(x)$ for $\mu = 17.2$



$$\eta(x) = -x \cdot \ln x$$

- For $y \in \mathcal{Y}$ and $u \in \mathcal{X}$, define the APP

$$\varphi_W(u|y) = \frac{W(y|u)}{\sum_{v \in \mathcal{X}} W(y|v)} .$$

- Two symbols $y_1, y_2 \in \mathcal{Y}$ are in the same bin, if

$$\mathcal{R}(\varphi_W(u|y_1)) = \mathcal{R}(\varphi_W(u|y_2)) , \quad \text{for all } u \in \mathcal{X} .$$

New APP

- Denote all the output letters in bin z as $\mathcal{B}(z)$
- Define the leading input of the bin as

$$u^* = \arg \max_{u \in \mathcal{X}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y) \right]$$

- Define a new APP measure by

$$\psi(u|z) = \min_{y \in \mathcal{B}(z)} \varphi_W(u|y) \quad \text{for all } u \neq u^*$$

and

$$\psi(u^*|z) = 1 - \sum_{u \neq u^*} \psi(u|z)$$

$$\alpha_u(y) \triangleq \frac{\psi(u|z)}{\varphi_W(u|y)} \cdot \frac{\varphi_W(u^*|y)}{\psi(u^*|z)}$$

Example

Consider a bin $\mathcal{B}(z) = \{y_1, y_2, y_3, y_4\}$,
where

	y_1	y_2	y_3	y_4
$\varphi_W(0 y)$	0.21	0.21	0.20	0.20
$\varphi_W(1 y)$	0.60	0.59	0.59	0.58
$\varphi_W(2 y)$	0.19	0.20	0.21	0.22

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- The leading input is $u^* = \arg \max_{u \in \{0,1,2\}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y) \right] =$

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$\varphi_W(0 y)$	0.21	0.21	0.20	0.20	$\psi(0 z) = 0.20$
$\varphi_W(1 y)$	0.60	0.59	0.59	0.58	
$\varphi_W(2 y)$	0.19	0.20	0.21	0.22	$\psi(2 z) = 0.19$

- The leading input is $u^* = \arg \max_{u \in \{0,1,2\}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y) \right] = 1$.

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$\varphi_W(0 y)$	0.21	0.21	0.20	0.20	$\psi(0 z) = 0.20$
$\varphi_W(1 y)$	0.60	0.59	0.59	0.58	$\psi(1 z) = 1 - (0.20 + 0.19) = 0.61$
$\varphi_W(2 y)$	0.19	0.20	0.21	0.22	$\psi(2 z) = 0.19$

- The leading input is $u^* = \arg \max_{u \in \{0,1,2\}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y) \right] = 1$.

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Consider a bin $\mathcal{B}(z) = \{y_1, y_2, y_3, y_4\}$,
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$\varphi_W(2 y)$	0.19	0.20	0.21	0.22	$\psi(2 z) = 0.19$

- The leading input is $u^* = \arg \max_{u \in \{0,1,2\}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y) \right] = 1$.
- E.g. $\alpha_0(y_1) = \frac{0.20}{0.21} \cdot \frac{0.60}{0.61} \approx 0.936$.

Thus, we subtract ($\approx 0.064 \cdot W(y_1|0)$) and pass over to our boost symbol κ_0 .

Proof Outline (very broadly)

Note that

Mutual Information of U and Y

$$I(W) = I(U; Y) = \ln q - \sum_{y \in \mathcal{Y}} p_W(y) \cdot H(U|Y = y)$$

Now, define

Surrogate for Mutual Information

$$\begin{aligned} J(U; Z) &= \ln q - \sum_{z \in \mathcal{Z}} p(z) \cdot H_\psi(U|Z = z) \\ &= \ln q - \sum_{z \in \mathcal{Z}} p(z) \sum_{u \in \mathcal{X}} \eta[\psi(u|z)] , \end{aligned}$$

where $p(z) \triangleq \sum_{y \in \mathcal{B}(z)} p_W(y)$.

Next, prove two bounds and sum up

- First, some interesting properties regarding our quantization, the APP measure $\psi(\cdot|z)$ and the leading input imply that

$$|\eta(\varphi_W(u|y)) - \eta(\psi(u|z))| \leq \begin{cases} 1/\mu & \text{if } u \neq u^* , \\ \frac{q-1}{\mu} & \text{if } u = u^* . \end{cases}$$

$$\Rightarrow J(U; Z) - I(W) = \sum p(y) [\eta(\varphi(u|y)) - \eta(\psi(u|z))] \leq 2 \cdot \frac{q-1}{\mu} ,$$

- Second,

$$\frac{1}{q} \sum_{u \in \mathcal{X}} \varepsilon_u \leq \frac{q(q-1)}{\mu} .$$

$$\begin{aligned} I(W') - J(U; Z) &= \\ &= \sum_z (p(z) - p_{W'}(z)) H_\psi(U|Z=z) - \sum_{\kappa} p_{W'}(\kappa) H(U|Z'=\kappa) \\ &\leq \ln q \cdot \frac{1}{q} \cdot \sum_{u \in \mathcal{X}} \varepsilon_u \leq \ln q \cdot \frac{q(q-1)}{\mu} . \end{aligned}$$