Bounds and Constructions for Granular Media Coding

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Motivation

- Conventional magnetic recording media: *grains*
- Wood *et al.* suggested a new approach for magnetizing areas as small as a single grain
- Mazumdar *et al.* considered a 1-D combinatorial error model with specific substitution errors and grains of length 1 and 2
- Overlaps vs. nonoverlapping grain patterns
- Related to shingled writing (Iyengar *et al.*)

Combinatorial model

- $[s] = \{0, 1, \ldots, s{-}1\}$
- An alphabet $\Sigma = [q]$
- A *grain* (of length 2) ending at location $e \in [n] \setminus \{0\}$ in a word $\boldsymbol{x} = (x_i)_{i \in [n]}$ smears the value of cell $e{-}1$ to cell e: $x_e \leftarrow x_{e-1}$
- A *grain pattern* $\mathcal{S} \subseteq [n] \setminus \{0\}$ contains all the grain locations and inflicts errors to $\boldsymbol{x}$ by means of operator $\sigma_{\mathcal{S}}$
- $\mathcal{S}$ has *overlaps* if there exist $e, e' \in \mathcal{S}$ such that $e' = e{+}1$; otherwise $\mathcal{S}$ is *nonoverlapping*

Example

- $\Sigma = [3]$, $n = 6$, $\boldsymbol{x} = 102022$, $\mathcal{S} = \{1, 3, 5\}$ and $\mathcal{S}' = \{1, 2\}$

$$
\begin{array}{rc}
\boldsymbol{x}: & \begin{array}{|c|c|c|c|c|c|}\hline 1 & 0 & 2 & 0 & 2 & 2 \\ \hline\end{array} \\
& \downarrow \\
\mathcal{S}: & \begin{array}{|c|c|c|c|c|c|}\hline 1 & 0 & 2 & 0 & 2 & 2 \\ \hline\end{array} \\
& \downarrow \\
\sigma_{\mathcal{S}}(\boldsymbol{x}): & \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & 2 \\ \hline\end{array}
\end{array}
\qquad
\begin{array}{rc}
\boldsymbol{x}: & \begin{array}{|c|c|c|c|c|c|}\hline 1 & 0 & 2 & 0 & 2 & 2 \\ \hline\end{array} \\
& \downarrow \\
\mathcal{S}': & \begin{array}{|c|c|c|c|c|c|}\hline 1 & 0 & 2 & 0 & 2 & 2 \\ \hline\end{array} \\
& \downarrow \\
\sigma_{\mathcal{S}'}(\boldsymbol{x}): & \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 0 & 0 & 2 & 2 \\ \hline\end{array}
\end{array}
$$

Combinatorial model (cont.)

- Words $\boldsymbol{x}, \boldsymbol{y} \in \Sigma^n$ are *t-confusable* if there exist $\mathcal{S}$, $\mathcal{S}'$ of size t at most for which $\sigma_{\mathcal{S}}(\boldsymbol{x}) = \sigma_{\mathcal{S}'}(\boldsymbol{y})$
- A code $\mathcal{C} \subseteq \Sigma^n$ is *t-grain-correcting* if no two distinct codewords are t-confusable
- Largest size $M_q(n,t)$, *rate* $R_q(\tau) = \limsup_{n\to\infty} \frac{1}{n} \log_q M_q(n,t)$
- Nonuniformity: 111 is 1-confusable with 3 words, whereas 101 is 1-confusable with 4 words
- Words $\mathbf{x}, \mathbf{x}' \in \Sigma_2^n$ are *t-confusable in the wide sense* (*t-cws*) if there exist $\mathcal{S}, \mathcal{S}'$ (possibly, with overlaps) such that

$$|\mathcal{S}| + |\mathcal{S}'| \leq 2t \text{ and } \sigma_{\mathcal{S}}(\mathbf{x}) = \sigma_{\mathcal{S}'}(\mathbf{x}')$$

Lower bound on $M_q(n,t)$

- The number $\mathsf{W}_t(\mathcal{X})$ of ordered pairs of t-confusable words in $\mathcal{X}$

Lemma. *Let n, t be positive integers and let $\mathcal{X} \subseteq \Sigma^n$, then*

$$M_q(n,t) \geq \frac{|\mathcal{X}|^2}{4\mathsf{W}_t(\mathcal{X})}.$$

- We will evaluate $\mathsf{W}_t(\mathcal{X})$ for certain sets $\mathcal{X}$ with prescribed empirical distribution of transitions

Graph $\mathcal{G}^{(\mathcal{N})} = (V^{(\mathcal{N})}, E^{(\mathcal{N})})$

- Set of states $V^{(\mathcal{N})} = V_0 \cup V_1 \cup V_2$ where $V_0 = \{aa : a \in \Sigma\}$, $V_1 = \left\{a\overline{b} : ab \in \Sigma^2, a \neq b\right\}$ and $V_2 = \left\{\overline{a}\overline{b} : ab \in \Sigma^2, a \neq b\right\}$
- For $q = 2$, $V_0 = \{00, 11\}$, $V_1 = \left\{\overline{0}1, 0\overline{1}, \overline{1}0, 1\overline{0}\right\}$, $V_2 = \left\{\overline{01}, \overline{10}\right\}$
- There is an edge in $E^{(\mathcal{N})}$ from v to v':

$$v: \boxed{0}\boxed{0} \xrightarrow{\text{[N1]:}} v': \boxed{1}\boxed{1}$$

$$v: \boxed{0}\boxed{0} \xrightarrow{\text{[N2]:}} v': \boxed{0}\boxed{\overline{1}}$$

$$v: \boxed{0}\boxed{\overline{1}} \xrightarrow{\text{[N3]:}} v': \boxed{\overline{2}}\boxed{0}$$

$$v: \boxed{0}\boxed{0} \xrightarrow{\text{[N4]:}} v': \boxed{\overline{1}}\boxed{\overline{2}}$$

Example

- $\Sigma = [3]$, $n = 6$, $\gamma = (v_i)_{i\in[n]} = 11 \quad 22 \quad 2\overline{0} \quad \overline{1}2 \quad 00 \quad \overline{12}$

$$\begin{array}{rc}
 & \boxed{1\,|\,1} \\
[\mathrm{N1}]: & \downarrow \\
 & \boxed{2\,|\,2} \\
[\mathrm{N2}]: & \downarrow \\
 & \boxed{2\,|\,\overline{0}} \\
[\mathrm{N3}]: & \downarrow \\
 & \boxed{\overline{1}\,|\,2} \\
[\mathrm{N1}]: & \downarrow \\
 & \boxed{0\,|\,0} \\
[\mathrm{N4}]: & \downarrow \\
 & \boxed{\overline{1}\,|\,\overline{2}}
\end{array}$$

- The patterns $\mathcal{S} = \{3, 5\}$ and $\mathcal{S}' = \{2, 5\}$ make $\boldsymbol{x} = 122101$ (the left path) and $\boldsymbol{y} = 120202$ (the right path) confusable

Adjacency matrix $A_{\mathcal{G}}^{(\mathcal{N})}$ for $q = 2$

$$A_{\mathcal{G}}^{(\mathcal{N})} = \begin{array}{c|cccccc} & 00 & \overline{0}1 & 0\overline{1} & \overline{1}0 & 1\overline{0} & 11 \\ \hline 00 & 1 & 0 & 1 & 1 & 0 & 1 \\ \overline{0}1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0\overline{1} & 1 & 0 & 0 & 1 & 0 & 1 \\ \overline{1}0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1\overline{0} & 1 & 1 & 0 & 0 & 0 & 1 \\ 11 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}$$

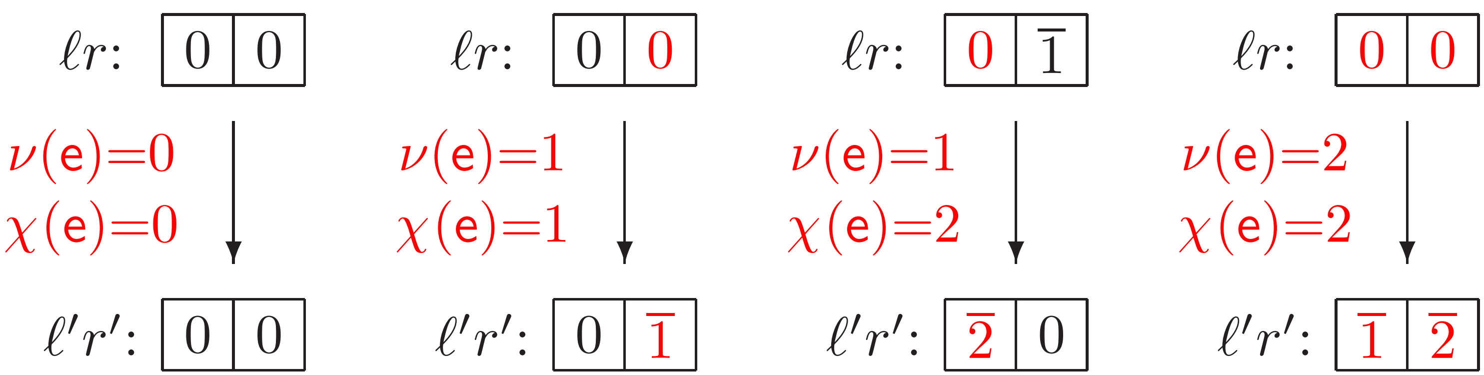
Correspondence between pairs of words and paths

Lemma. *For each t-cws (ordered) pair $(\boldsymbol{x}, \boldsymbol{y}) \in \Sigma^n \times \Sigma^n$ there is at least one path $\gamma = (v_i)_{i \in [n]} = (\ell_i r_i)_{i \in [n]}$ in $\mathcal{G}^{(\mathcal{N})}$ such that*

1. $v_0 \in V_0$,
2. $\boldsymbol{x} = (\partial(\ell_i))_{i \in [n]}$,
3. $\boldsymbol{y} = (\partial(r_i))_{i \in [n]}$ *and*
4. *The total number of bars in γ is at most $2t$.*

Function $f^{(\mathcal{N})}$

- Function $f^{(\mathcal{N})} : E^{(\mathcal{N})} \to [3]^2$
- For an edge $\mathsf{e} = (\ell r, \ell' r') \in E^{(\mathcal{N})}$, $f^{(\mathcal{N})}(\mathsf{e}) = (\nu(\mathsf{e}), \chi(\mathsf{e}))$
- $\nu(\mathsf{e})$ counts the smallest number of grains confusing $\ell\ell'$ and rr'; $\chi(\mathsf{e})$ counts the number of crossovers



Matrix function $\mathsf{A}_{\mathcal{G}}^{(\mathcal{N})}$

- $\left[\mathsf{A}_{\mathcal{G}}^{(\mathcal{N})}(z,h)\right]_{v,v'\in V} = \begin{cases} z^{\nu(\mathsf{e})}h^{\chi(\mathsf{e})} & \mathsf{e}=(v,v')\in E^{(\mathcal{N})} \\ 0 & \text{otherwise} \end{cases}$
- For $q=2$,

$$\mathsf{A}_{\mathcal{G}}^{(\mathcal{N})}(z,h) = \begin{array}{c|cccccc} & 00 & \bar{0}1 & 0\bar{1} & \bar{1}0 & 1\bar{0} & 11 \\ \hline 00 & 1 & 0 & hz & hz & 0 & h^2 \\ \bar{0}1 & h & 0 & 0 & 0 & h^2z & h \\ 0\bar{1} & h & 0 & 0 & h^2z & 0 & h \\ \bar{1}0 & h & 0 & h^2z & 0 & 0 & h \\ 1\bar{0} & h & h^2z & 0 & 0 & 0 & h \\ 11 & h^2 & hz & 0 & 0 & hz & 1 \end{array}$$

Main theorem

- Applying special cases of lemmas from [MR92]: optimizing convex functions subject to linear equality and inequality constraints
- Asymptotic upper bound on the number of paths with average number of crossovers $\sim 2p$ and number of confusing grains $\leq 2\tau$

$$K^{(\mathcal{N})} = \inf_{z\in(0,1],\, h\in(0,\infty)} \{\log_q \lambda(\mathsf{A}_{\mathcal{G}}^{(\mathcal{N})}(z,h)) - 2\tau \log_q z - 2p \log_q h\}$$

Theorem. *Let* $\tau \in (0,1)$, *then*[a]

$$R_q(\tau) \geq \varrho_q^{(\mathcal{N})}(\tau) = \sup_{p\in[0,1]} \left\{ 2\mathsf{H}_q(p) - K^{(\mathcal{N})} \right\}$$

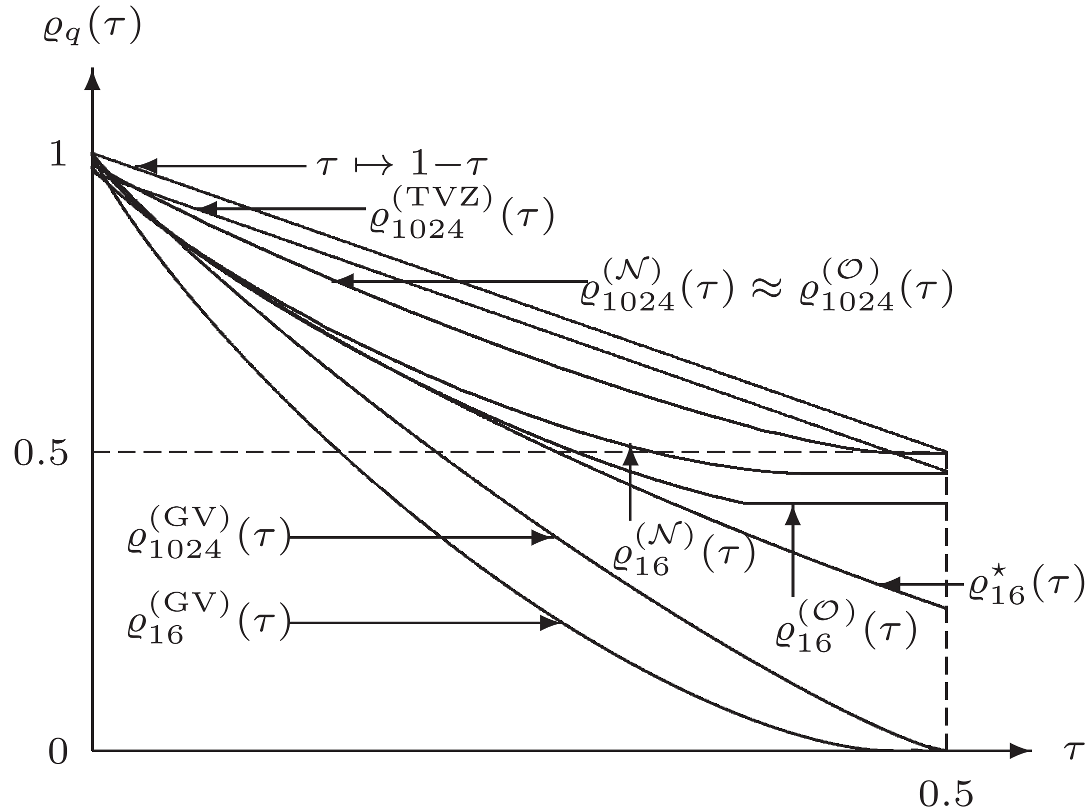
[a] Asymptotic version of $M_q(n,t) \geq \frac{|\mathcal{X}|^2}{4\mathsf{W}_t(\mathcal{X})}$

Merging states of $\mathcal{G}^{(\mathcal{N})}$

- Similar to the standard procedure for reducing the number of states in a presentation of a constrained system while preserving its spectral radius
- The states of V_0 can be merged into superstate 0, V_1 — into superstate 1, V_2 — into superstate 2
- Reduced matrix $\mathcal{A}_{\mathcal{G}}^{(\mathcal{N})}$:

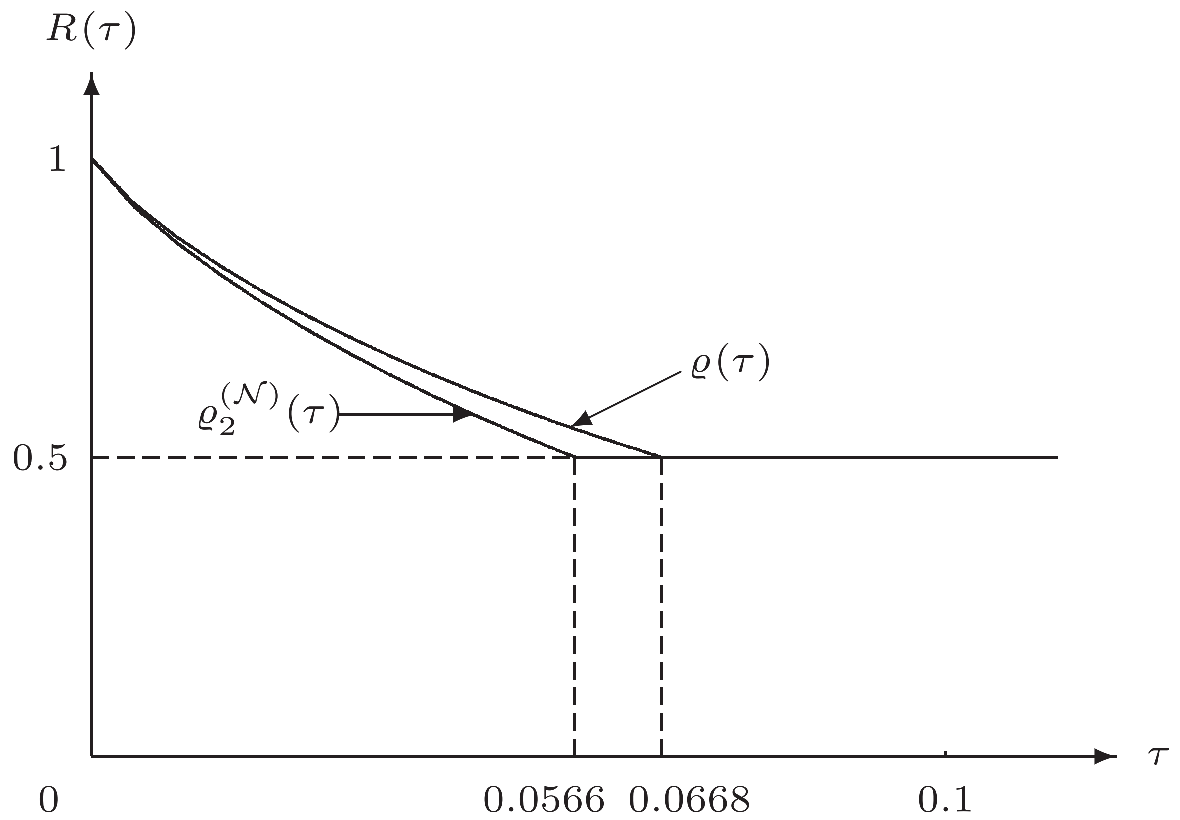
$$\mathcal{A}_{\mathcal{G}}^{(\mathcal{N})} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1+(q-1)h^2 & 2(q-1)hz & (q-1)(q-2)h^2z^2 \\ 1 & 2h+(q-2)h^2 & (q-1)h^2z & 0 \\ 2 & 2h+(q-2)h^2 & 0 & 0 \end{array}$$

Lower bounds $\varrho_q^{(j)}(\tau)$



Bounds $\varrho_{1024}^{(\mathrm{TVZ})}(\tau)$ and $\varrho_{16}^{\star}(\tau)$ rely on the code
$\mathcal{C}^\star = \{\boldsymbol{c} = (c_i)_{i\in[n]} \in \Sigma^n : c_i \neq c_{i+1}$ for any $i \in [n-1]\}$

Improved lower bound for $q = 2$



Based on the idea of Gabrys *et al.*

Comb. upper bound on $M(n,t)$

- $\mathcal{C}$ is a t-grain-correcting code of length n
- There exists a subcode $\mathcal{C}'$ of $\mathcal{C}$ of size $|\mathcal{C}|/n$ at least whose codewords have the same number of runs r
- Set $\Phi_t(\mathbf{x})$ of all words in Σ_2^n which are t-confusable with $\mathbf{x}$
- Lower bound $\psi_t(r)$ on the size of $\Phi_t(\mathbf{x})$ depending only on the number of runs
- $|\mathcal{C}'| \le \frac{2}{\psi_t(r)} \cdot \sum_{i=\max\{0,r-2t\}}^{r} \binom{n-1}{i-1}$
- $|\mathcal{C}| \le \frac{2n}{\psi_t(r)} \cdot \sum_{i=\max\{0,r-2t\}}^{r} \binom{n-1}{i-1}$
- $M(n,t) \le \max_{r=1}^{n} \left\{ \frac{2n}{\psi_t(r)} \cdot \sum_{i=\max\{0,r-2t\}}^{r} \binom{n-1}{i-1} \right\}$

Upper bound on $R(\tau)$

Corollary. *Let* $\tau \in [0, 0.5]$. *Then*

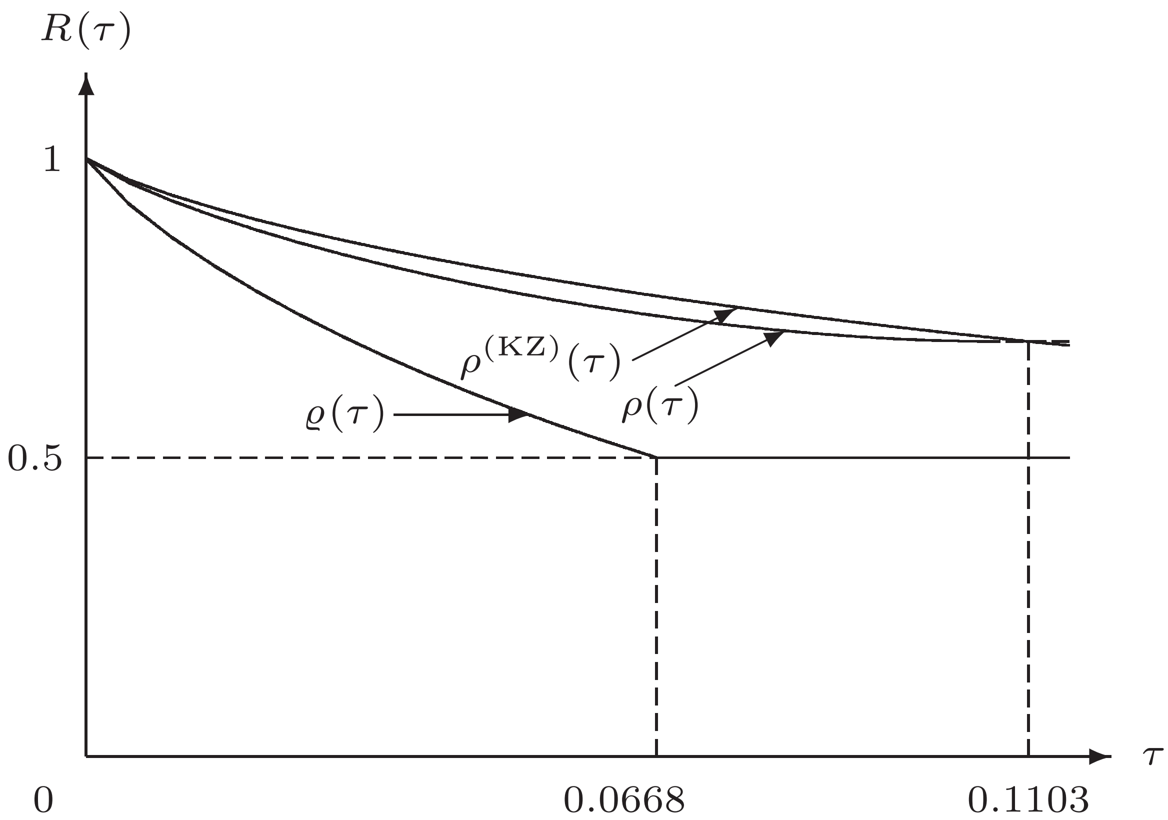
$$R(\tau) \leq \sup_{\rho \in [0,0.5]} \left\{ H(\rho) - \varliminf_{n \to \infty} \frac{1}{n} \log_2 \psi_{\lceil \tau n \rceil}(\lceil \rho n \rceil) \right\} .$$

- $\varliminf_{n \to \infty} \frac{1}{n} \log_2 \psi_{\lceil \tau n \rceil}(\lceil \rho n \rceil) \geq (\rho - \tau) H\left(\frac{\tau}{\rho - \tau}\right)$ [Kashyap and Zémor]

Theorem. *Let* $\tau \in [0, 0.5]$. *Then*

$$R(\tau) \leq \rho(\tau) = \sup_{\rho \in [2\tau, 0.5]} \left\{ H(\rho) - (\rho - \tau) H\left(\frac{\tau}{\rho - \tau}\right) \right\} .$$

Comparison of upper bounds



Constructions of 1-grain-correcting codes

- Well-known partitioning technique (Al-Bassam *et al.*)
- Words $\mathbf{x}, \mathbf{x}' \in \Sigma_2^n$ are 1-*strongly-confusable* if either $0\mathbf{x} \in \Sigma_2^{n+1}$ and $0\mathbf{x}' \in \Sigma_2^{n+1}$ are 1-confusable or $1\mathbf{x} \in \Sigma_2^{n+1}$ and $1\mathbf{x}' \in \Sigma_2^{n+1}$ are 1-confusable
- Set $\Sigma_{2,\text{even}}^{\beta} = \left\{ \mathbf{x} \in \Sigma_2^{\beta} : \mathsf{w}\left(\mathbf{x}\right) \text{ is even} \right\}$
- Sets $\mathcal{X}_i$, $i \in \alpha^{\star}$, form a partition of Σ_2^{α} such that each set $\mathcal{X}_i$ is a 1-grain-correcting code in the strong sense
- Sets $\mathcal{Y}_i$, $i \in \beta^{\star}$, form a partition of $\Sigma_{2,\text{even}}^{\beta}$ such that each set $\mathcal{Y}_i$ is a 1-grain-correcting code in the strong sense

1-grain-correcting code

Theorem. *The following code of length* $n = \alpha+\beta+1$,

$$\mathcal{C} = \bigcup_{i \in \langle \min\{\alpha^\star,\beta^\star\}\rangle} \Sigma_2 \times \mathcal{X}_i \times \mathcal{Y}_i \ ,$$

is a 1*-grain-correcting code of size* $2\sum_{i\in\langle\min\{\alpha^\star,\beta^\star\}\rangle} |\mathcal{X}_i| \cdot |\mathcal{Y}_i|$.

Proof: Let $\boldsymbol{c} = (z\ \mathbf{x}\ \mathbf{y}), \boldsymbol{c}' = (z\ \mathbf{x}'\ \mathbf{y}')$ be distinct codewords in $\mathcal{C}$ such that $z \in \Sigma_2$, $\mathbf{x} \in \mathcal{X}_i$, $\mathbf{x}' \in \mathcal{X}_{i'}$, $\mathbf{y} \in \mathcal{Y}_i$, $\mathbf{y}' \in \mathcal{Y}_{i'}$.

- $i{=}i'$, $\mathbf{x}{\neq}\mathbf{x}'$ then $\boldsymbol{c}$, $\boldsymbol{c}'$ not 1-confusable due to $z\mathbf{x}$ and $z\mathbf{x}'$
- $i{=}i'$, $\mathbf{y}{\neq}\mathbf{y}'$ then $\boldsymbol{c}$, $\boldsymbol{c}'$ not 1-confusable due to $x_{\alpha-1}\mathbf{y}$ and $x'_{\alpha-1}\mathbf{y}'$
- $i{\neq}i'$, then $\mathsf{d}_H(\mathbf{x},\mathbf{x}') \geq 1$ and $\mathsf{d}_H(\mathbf{y},\mathbf{y}') \geq 2$ $\square$

How to obtain $\mathcal{X}_i$ and $\mathcal{Y}_i$

- *Nonconfusability graph* $\mathcal{G}_n(\mathcal{Z}) = (\mathcal{Z}, E)$, where $\mathcal{Z} \subseteq \Sigma_2^n$

INPUT: graph $\mathcal{G} = (V, E)$;
$\boldsymbol{\pi} \leftarrow \emptyset$; // $\boldsymbol{\pi}$ is a partition of V
while $V \neq \emptyset$ **do** {
    $\mathcal{B} \leftarrow$ MAXIMUMCLIQUE($\mathcal{G}$);
    $\mathcal{G} \leftarrow \mathcal{G} \setminus \mathcal{B}$; // remove all the states of $\mathcal{B}$ from V and
    // all the edges connected to $\mathcal{B}$ from E
    ADD($\boldsymbol{\pi}, \mathcal{B}$); // append $\mathcal{B}$ to the end of the list $\boldsymbol{\pi}$
}

- The sets $\mathcal{X}_i$ obtained by applying the procedure to $\mathcal{G} = \mathcal{G}_\alpha(\Sigma_2^\alpha)$, the sets $\mathcal{Y}_i$ obtained by applying it to $\mathcal{G} = \mathcal{G}_\beta(\Sigma_{2,\text{even}}^\beta)$

Best known bounds on $M(n,1)$

n	2	3	4	5	6	7	8	9	10	11
Lower bound	**2**	**4**	**6**	**8**	**16**	**26**	**44**	72	*112*	*206*
Upper bound	**2**	**4**	**6**	**8**	**16**	**26**	**44**	88	176	352

n	12	13	14	15	16	17	18
Lower bound	*372*	*686*	*1272*	*2384*	*4522*	*8428*	*15348*
Upper bound	682	1260	2340	4368	8192	15420	29126