Bounds and Constructions for Granular Media Coding

Artyom Sharov, Ronny Roth

Department of Computer Science
Technion – Israel Institute of Technology
Motivation

- Conventional magnetic recording media: grains
- Wood et al. suggested a new approach for magnetizing areas as small as a single grain
- Mazumdar et al. considered a 1-D combinatorial error model with specific substitution errors and grains of length 1 and 2
- Overlaps vs. nonoverlapping grain patterns
- Related to shingled writing (Iyengar et al.)
Combinatorial model

- \([s] = \{0, 1, \ldots, s-1\}\)
- An alphabet \(\Sigma = [q]\)
- A grain (of length 2) ending at location \(e \in [n]\setminus \{0\}\) in a word \(x = (x_i)_{i \in [n]}\) smears the value of cell \(e-1\) to cell \(e\): \(x_e \leftarrow x_{e-1}\)
- A grain pattern \(S \subseteq [n]\setminus \{0\}\) contains all the grain locations and inflicts errors to \(x\) by means of operator \(\sigma_S\)
- \(S\) has overlaps if there exist \(e, e' \in S\) such that \(e' = e+1\); otherwise \(S\) is nonoverlapping
Example

- $\Sigma = [3]$, $n = 6$, $x = 102022$, $S = \{1, 3, 5\}$ and $S' = \{1, 2\}$

$x: \begin{array}{cccccc}
1 & 0 & 2 & 0 & 2 & 2
\end{array}$

$S: \begin{array}{cccccc}
1 & 0 & 2 & 0 & 2 & 2
\end{array}$

$\sigma_S(x): \begin{array}{cccccc}
1 & 1 & 2 & 2 & 2 & 2
\end{array}$

$x: \begin{array}{cccccc}
1 & 0 & 2 & 0 & 2 & 2
\end{array}$

$S': \begin{array}{cccccc}
1 & 0 & 2 & 0 & 2 & 2
\end{array}$

$\sigma_{S'}(x): \begin{array}{cccccc}
1 & 1 & 0 & 0 & 2 & 2
\end{array}$
Combinatorial model (cont.)

• Words \( x, y \in \Sigma^n \) are \( t \)-confusable if there exist \( S, S' \) of size \( t \) at most for which \( \sigma_S(x) = \sigma_{S'}(y) \)

• A code \( C \subseteq \Sigma^n \) is \( t \)-grain-correcting if no two distinct codewords are \( t \)-confusable

• Largest size \( M_q(n, t) \), rate \( R_q(\tau) = \limsup_{n \to \infty} \frac{1}{n} \log_q M_q(n, t) \)

• Nonuniformity: 111 is 1-confusable with 3 words, whereas 101 is 1-confusable with 4 words

• Words \( x, x' \in \Sigma_2^n \) are \( t \)-confusable in the wide sense (\( t \)-cws) if there exist \( S, S' \) (possibly, with overlaps) such that

\[
|S| + |S'| \leq 2t \text{ and } \sigma_S(x) = \sigma_{S'}(x')
\]
Lower bound on $M_q(n, t)$

- The number $W_t(X)$ of ordered pairs of $t$-confusable words in $X$

Lemma. Let $n$, $t$ be positive integers and let $X \subseteq \Sigma^n$, then

$$M_q(n, t) \geq \frac{|X|^2}{4W_t(X)}.$$  

- We will evaluate $W_t(X)$ for certain sets $X$ with prescribed empirical distribution of transitions
Graph $G^{(N)} = (V^{(N)}, E^{(N)})$

- Set of states $V^{(N)} = V_0 \cup V_1 \cup V_2$ where $V_0 = \{aa : a \in \Sigma\}$, $V_1 = \{a\overline{b} : ab \in \Sigma^2, a \neq b\}$ and $V_2 = \{\overline{a}b : ab \in \Sigma^2, a \neq b\}$
- For $q = 2$, $V_0 = \{00, 11\}$, $V_1 = \{0\overline{1}, 0\overline{1}, \overline{1}0, \overline{1}0\}$, $V_2 = \{0\overline{1}, \overline{1}0\}$
- There is an edge in $E^{(N)}$ from $v$ to $v'$:

\[
\begin{align*}
v &: 0 \ 0 \\
[N1]: & \downarrow \\
v' &: 1 \ 1 \\

v &: 0 \ 0 \\
[N2]: & \downarrow \\
v' &: 0 \ \overline{1} \\

v &: 0 \ \overline{1} \\
[N3]: & \downarrow \\
v' &: \overline{2} \ 0 \\

v &: 0 \ 0 \\
[N4]: & \downarrow \\
v' &: \overline{1} \ \overline{2}
\end{align*}
\]
Example

- \( \Sigma = [3], n = 6, \gamma = (v_i)_{i \in [n]} = 11 \quad 22 \quad 20 \quad 12 \quad 00 \quad 12 \)

\[
\begin{array}{cccccc}
1 & 1 & 1 \\
N1: & \downarrow & 2 & 2 \\
N2: & \downarrow & 2 & 1 \\
N3: & \downarrow & 1 & 2 \\
N1: & \downarrow & 0 & 0 \\
N4: & \downarrow & 1 & 2 \\
\end{array}
\]

- The patterns \( S = \{3, 5\} \) and \( S' = \{2, 5\} \) make \( x = 122101 \) (the left path) and \( y = 120202 \) (the right path) confusable.
Adjacency matrix $A_G^{(N)}$ for $q = 2$

$$
A_G^{(N)} =
\begin{array}{|ccccccc|}
\hline
 & 00 & \bar{0}1 & 0\bar{1} & \bar{1}0 & 1\bar{0} & 11 \\
\hline
00 & 1 & 0 & 1 & 1 & 0 & 1 \\
\bar{0}1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0\bar{1} & 1 & 0 & 0 & 1 & 0 & 1 \\
\bar{1}0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1\bar{0} & 1 & 1 & 0 & 0 & 0 & 1 \\
11 & 1 & 1 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$
**Correspondence between pairs of words and paths**

**Lemma.** For each $t$-cws (ordered) pair $(x, y) \in \Sigma^n \times \Sigma^n$ there is at least one path $\gamma = (v_i)_{i \in [n]} = (\ell_i r_i)_{i \in [n]}$ in $G^{(N)}$ such that

1. $v_0 \in V_0$,
2. $x = (\partial(\ell_i))_{i \in [n]}$,
3. $y = (\partial(r_i))_{i \in [n]}$ and
4. The total number of bars in $\gamma$ is at most $2t$. 
• Function \( f^{(N)} : E^{(N)} \rightarrow [3]^2 \)

• For an edge \( e = (\ell r, \ell' r') \in E^{(N)} \), \( f^{(N)}(e) = (\nu(e), \chi(e)) \)

• \( \nu(e) \) counts the smallest number of grains confusing \( \ell \ell' \) and \( rr' \);
  \( \chi(e) \) counts the number of crossovers

<table>
<thead>
<tr>
<th>( \ell r ):</th>
<th>0 0</th>
<th>0 0</th>
<th>0 1</th>
<th>0 0</th>
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<tr>
<td>( \nu(e)=0 )</td>
<td>( \nu(e)=1 )</td>
<td>( \nu(e)=1 )</td>
<td>( \nu(e)=2 )</td>
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<tr>
<td>( \chi(e)=0 )</td>
<td>( \chi(e)=1 )</td>
<td>( \chi(e)=2 )</td>
<td>( \chi(e)=2 )</td>
<td></td>
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</tbody>
</table>

| \( \ell' r' \): | 0 0 | 0 1 | 2 0 | 1 2 |
Matrix function $A_{G}^{(\mathcal{N})}$

- $[A_{G}^{(\mathcal{N})}(z, h)]_{v, v' \in V} = \begin{cases} 
  z^{\nu(e)} h^{\chi(e)} & e = (v, v') \in E^{(\mathcal{N})} \\
  0 & \text{otherwise}
\end{cases}$

- For $q = 2$,

$$
\begin{array}{c|cccccc}
 & 00 & \bar{0}1 & 0\bar{1} & \bar{1}0 & 1\bar{0} & 11 \\
\hline
00 & 1 & 0 & hz & hz & 0 & h^2 \\
\bar{0}1 & h & 0 & 0 & 0 & h^2z & h \\
0\bar{1} & h & 0 & 0 & h^2z & 0 & h \\
\bar{1}0 & h & 0 & h^2z & 0 & 0 & h \\
1\bar{0} & h & h^2z & 0 & 0 & 0 & h \\
11 & h^2 & hz & 0 & 0 & hz & 1
\end{array}
$$
Main theorem

• Applying special cases of lemmas from [MR92]: optimizing convex functions subject to linear equality and inequality constraints

• Asymptotic upper bound on the number of paths with average number of crossovers $\sim 2p$ and number of confusing grains $\leq 2\tau$

$$K^{(N)} = \inf_{z \in (0,1), h \in (0,\infty)} \{ \log_q \lambda(A_g^{(N)}(z, h)) - 2\tau \log_q z - 2p \log_q h \}$$

**Theorem.** Let $\tau \in (0, 1)$, then\(^a\)

$$R_q(\tau) \geq \rho_q^{(N)}(\tau) = \sup_{p \in [0,1]} \left\{ 2H_q(p) - K^{(N)} \right\}$$

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\(^a\)Asymptotic version of $M_q(n, t) \geq \frac{|x|^2}{4W_t(x)}$
Merging states of $G^{(N)}$

- Similar to the standard procedure for reducing the number of states in a presentation of a constrained system while preserving its spectral radius.

- The states of $V_0$ can be merged into superstate 0, $V_1$ — into superstate 1, $V_2$ — into superstate 2.

- Reduced matrix $A_{G}^{(N)}$:

$$A_{G}^{(N)} = \begin{array}{ccc}
0 & 1 & 2 \\
0 & 1+(q-1)h^2 & 2(q-1)hz & (q-1)(q-2)h^2z^2 \\
1 & 2h+(q-2)h^2 & (q-1)h^2z & 0 \\
2 & 2h+(q-2)h^2 & 0 & 0 \\
\end{array}$$
Lower bounds $q_{q}^{(j)}(\tau)$

Bounds $q_{1024}^{(TVZ)}(\tau)$ and $q_{16}^{*}(\tau)$ rely on the code $C^* = \{ c = (c_i)_{i \in [n]} \in \Sigma^n : c_i \neq c_{i+1} \text{ for any } i \in [n-1] \}$
Improved lower bound for $q = 2$

Based on the idea of Gabrys et al.
Comb. upper bound on $M(n, t)$

- $\mathcal{C}$ is a $t$-grain-correcting code of length $n$
- There exists a subcode $\mathcal{C}'$ of $\mathcal{C}$ of size $|\mathcal{C}| / n$ at least whose codewords have the same number of runs $r$
- Set $\Phi_t(x)$ of all words in $\Sigma_2^n$ which are $t$-confusable with $x$
- Lower bound $\psi_t(r)$ on the size of $\Phi_t(x)$ depending only on the number of runs

- $|\mathcal{C}'| \leq \frac{2}{\psi_t(r)} \cdot \sum_{i=\max\{0, r-2t\}}^{r} \binom{n-1}{i-1}$
- $|\mathcal{C}| \leq \frac{2n}{\psi_t(r)} \cdot \sum_{i=\max\{0, r-2t\}}^{r} \binom{n-1}{i-1}$
- $M(n, t) \leq \max_{r=1}^{n} \left\{ \frac{2n}{\psi_t(r)} \cdot \sum_{i=\max\{0, r-2t\}}^{r} \binom{n-1}{i-1} \right\}$
Upper bound on $R(\tau)$

**Corollary.** Let $\tau \in [0, 0.5]$. Then

$$R(\tau) \leq \sup_{\rho \in [0, 0.5]} \left\{ H(\rho) - \lim_{n \to \infty} \frac{1}{n} \log_2 \psi_{\lceil \tau n \rceil}(\lceil \rho n \rceil) \right\}.$$

- $\lim_{n \to \infty} \frac{1}{n} \log_2 \psi_{\lceil \tau n \rceil}(\lceil \rho n \rceil) \geq (\rho - \tau)H\left(\frac{\tau}{\rho - \tau}\right)$ [Kashyap and Zémor]

**Theorem.** Let $\tau \in [0, 0.5]$. Then

$$R(\tau) \leq \rho(\tau) = \sup_{\rho \in [2\tau, 0.5]} \left\{ H(\rho) - (\rho - \tau)H\left(\frac{\tau}{\rho - \tau}\right) \right\}.$$
Comparison of upper bounds

\[ R(\tau) \]

- \( \tau = 0.0668 \)
- \( \tau = 0.1103 \)

- \( \rho^{(KZ)}(\tau) \)
- \( \rho(\tau) \)
- \( g(\tau) \)
Constructions of 1-grain-correcting codes

- Well-known partitioning technique (Al-Bassam et al.)
- Words $x, x' \in \Sigma_2^n$ are 1-*strongly-confusable* if either $0x \in \Sigma_2^{n+1}$ and $0x' \in \Sigma_2^{n+1}$ are 1-confusable or $1x \in \Sigma_2^{n+1}$ and $1x' \in \Sigma_2^{n+1}$ are 1-confusable
- Set $\Sigma_2^{\beta, \text{even}} = \{x \in \Sigma_2^\beta : w(x) \text{ is even}\}$
- Sets $\mathcal{X}_i, i \in \alpha^*$, form a partition of $\Sigma_2^\alpha$ such that each set $\mathcal{X}_i$ is a 1-grain-correcting code in the strong sense
- Sets $\mathcal{Y}_i, i \in \beta^*$, form a partition of $\Sigma_2^{\beta, \text{even}}$ such that each set $\mathcal{Y}_i$ is a 1-grain-correcting code in the strong sense
1-grain-correcting code

Theorem. The following code of length $n = \alpha + \beta + 1$,

$$
C = \bigcup_{i \in \langle \min\{\alpha^*, \beta^*\} \rangle} \Sigma_2 \times X_i \times Y_i,
$$

is a 1-grain-correcting code of size $2 \sum_{i \in \langle \min\{\alpha^*, \beta^*\} \rangle} |X_i| \cdot |Y_i|$.

Proof: Let $c = (z \ x \ y), c' = (z \ x' \ y')$ be distinct codewords in $C$ such that $z \in \Sigma_2, x \in X_i, x' \in X_i', y \in Y_i, y' \in Y_i'$.

- $i = i', x \neq x'$ then $c, c'$ not 1-confusable due to $zx$ and $zx'$
- $i = i', y \neq y'$ then $c, c'$ not 1-confusable due to $x_{\alpha-1}y$ and $x'_{\alpha-1}y'$
- $i \neq i'$, then $d_H(x, x') \geq 1$ and $d_H(y, y') \geq 2$ □
How to obtain $X_i$ and $Y_i$

- Nonconfusability graph $G_n(\mathcal{Z}) = (\mathcal{Z}, E)$, where $\mathcal{Z} \subseteq \Sigma_2^n$

```
INPUT: graph $\mathcal{G} = (V, E)$;
\pi \leftarrow \emptyset; \quad // \pi$ is a partition of $V$
while $V \neq \emptyset$ do {
    $\mathcal{B} \leftarrow \text{MaximumClique}(\mathcal{G})$
    $\mathcal{G} \leftarrow \mathcal{G} \setminus \mathcal{B}$; \quad // remove all the states of $\mathcal{B}$ from $V$ and
    \quad // all the edges connected to $\mathcal{B}$ from $E$
    $\text{ADD}(\pi, \mathcal{B})$; \quad // append $\mathcal{B}$ to the end of the list $\pi$
}
```

- The sets $X_i$ obtained by applying the procedure to $\mathcal{G} = \mathcal{G}_\alpha(\Sigma_2^\alpha)$, the sets $Y_i$ obtained by applying it to $\mathcal{G} = \mathcal{G}_\beta(\Sigma_2^{\text{even}})$
### Best known bounds on $M(n, 1)$

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<th>4</th>
<th>5</th>
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<tr>
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<td>4</td>
<td>6</td>
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