

Constrained Codes for Rank Modulation

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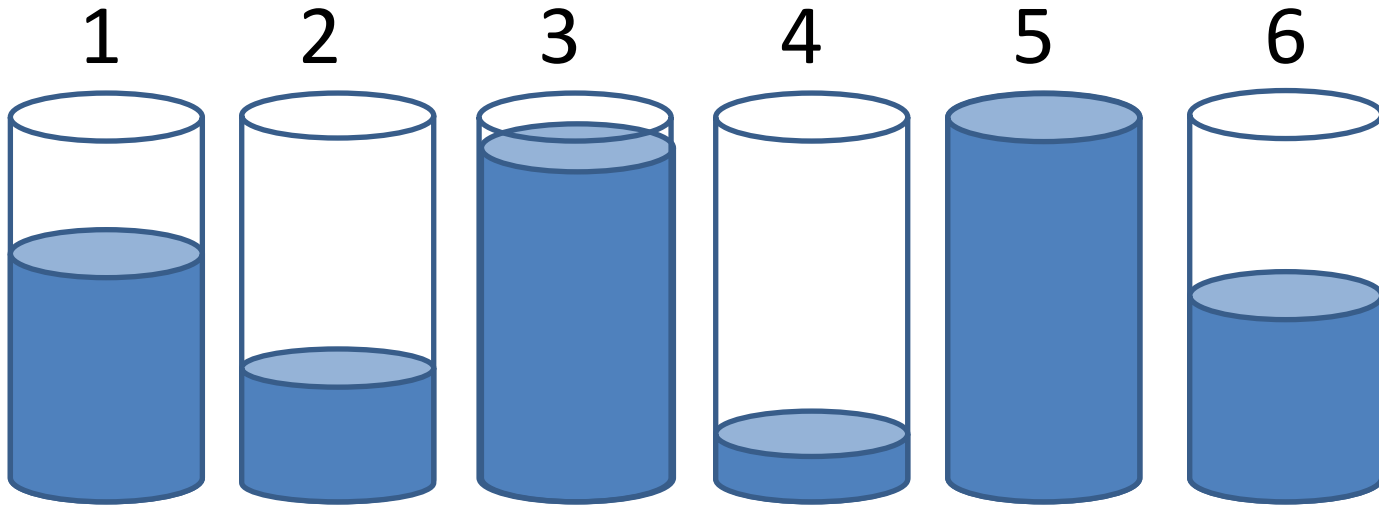
Coding Theory Seminar

May 25, 2014

Outline

- Introduction
- Two-neighbor constraint.
- Constrained error-correcting codes.
- Asymmetric two-neighbor constraint.

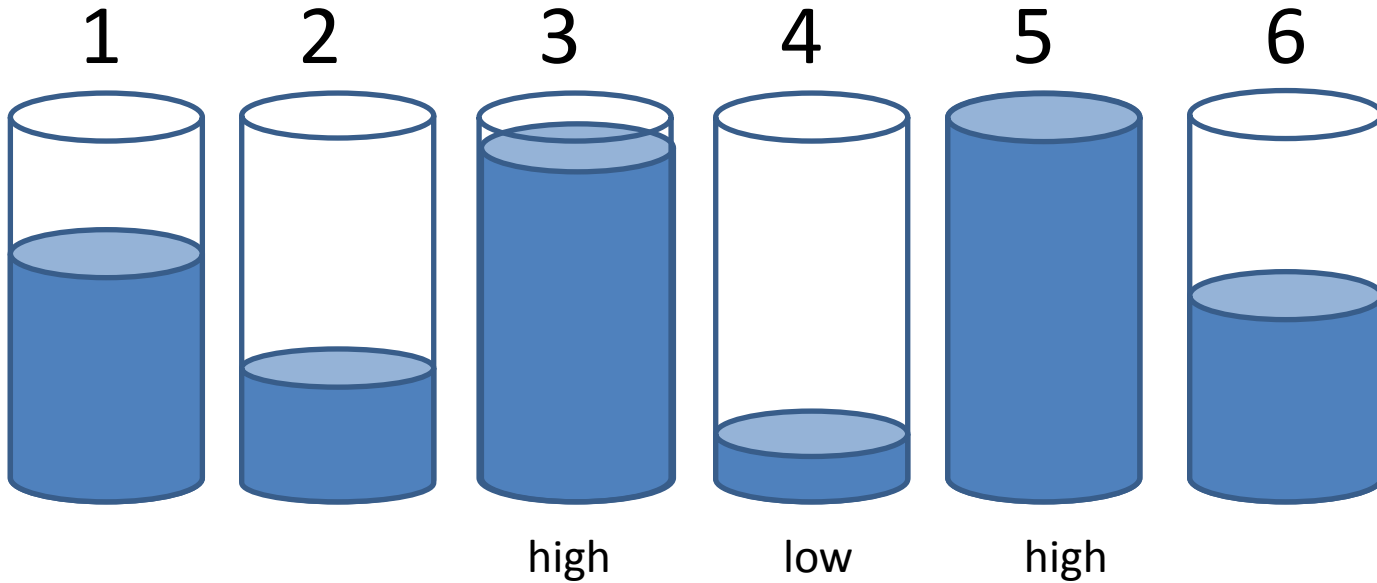
Rank Modulation for Flash Memory



[4,2,5,1,6,3]

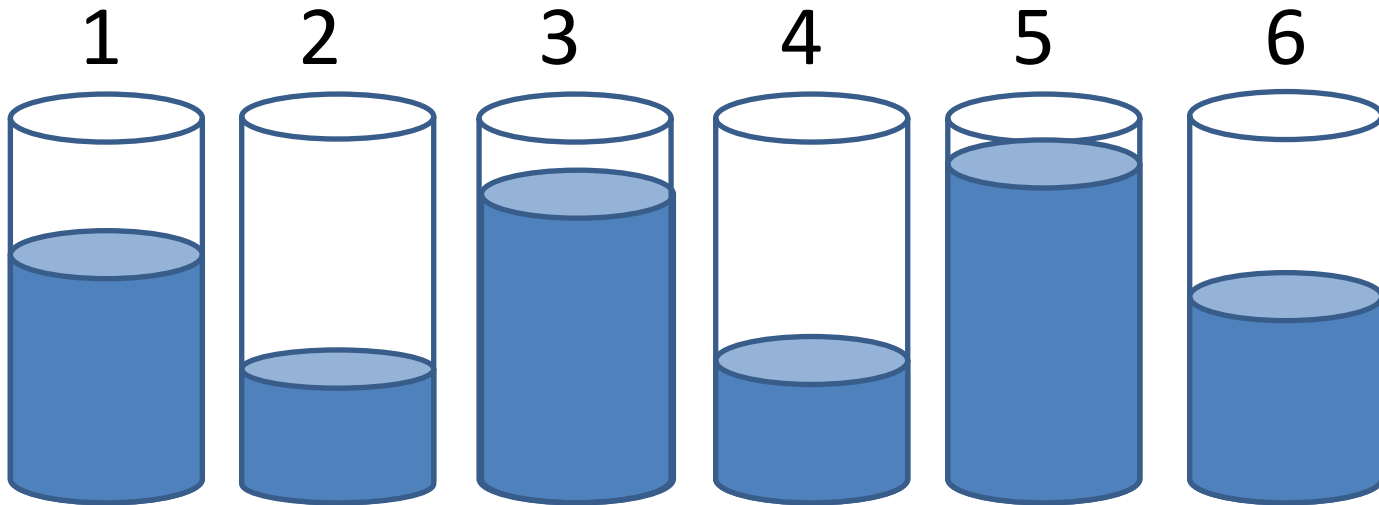
Codewords are permutations!

Inter-Cell Interference



[4,2,5,1,6,3]
high low high

Inter-Cell Interference



[4,1,5,2,6,3]

Single-Neighbor-Constraint

- S_n -the set of all permutations on n -elements.
- $\sigma \in S_n$ satisfies the **single-neighbor k -constraint** if for every $1 \leq i \leq n - 1$, $|\sigma(i) - \sigma(i + 1)| \leq k$.

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Example:

$[5,6,4,2,1,3]$ satisfies the single-neighbor 2-constraint.

Single-Neighbor Constraint

- Let $A^{SC}(n, k) \subset S_n$ consists of all permutations that satisfy the single-neighbor k -constraint.
- Let $C^{SC}(\epsilon) = \lim_{n \rightarrow \infty} \frac{\log |A^{SC}(n, k)|}{\log n!}$, $k = n^\epsilon$.

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Sala & Dolecek 13: $C^{SC}(\epsilon) = \epsilon$, for all $0 \leq \epsilon \leq 1$.

Two-Neighbor Constraint

- $\sigma \in S_n$ satisfies the **two-neighbor k -constraint** if for every $2 \leq i \leq n - 1$, either $|\sigma(i) - \sigma(i - 1)| \leq k$ or $|\sigma(i) - \sigma(i + 1)| \leq k$.

Two-Neighbor Constraint

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Example:

$[6, 1, 2, 5, 4, 3]$ satisfies the two-neighbor 1-constraint.

Two-Neighbor Constraint

- Let $A(n, k) \subset S_n$ consists of all permutations that satisfy the two-neighbor k -constraint.
- Let $C(\epsilon) = \lim_{n \rightarrow \infty} \frac{\log |A(n, k)|}{\log n!}$, $k = n^\epsilon$.

Two-Neighbor Constraint

- Let $A(n, k) \subset S_n$ consists of all permutations that satisfy the two-neighbor k -constraint.
- Let $C(\epsilon) = \lim_{n \rightarrow \infty} \frac{\log |A(n, k)|}{\log n!}$, $k = n^\epsilon$.

Theorem: For all $0 \leq \epsilon \leq 1$, $C(\epsilon) = \frac{1+\epsilon}{2}$.

Lower Bound on $|A(n, k)|$

- Let $P_{\ell, m}$ be the set of all **multi-permutations** on
on

$$\{ \underbrace{1, 1, \dots, 1}_{m \text{ times}}, \underbrace{2, 2, \dots, 2}_{m \text{ times}}, \dots, \underbrace{\ell, \ell, \dots, \ell}_{m \text{ times}} \}.$$

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Example: $\ell = 3, m = 2,$

$$\rho = [3, 1, 2, 2, 1, 3] \in P_{\ell, m}.$$

Lower Bound on $|A(n, k)|$

- Let $DP_{\ell, m}$ be the set

$$\{[\rho(1), \rho(1), \rho(2), \rho(2), \dots, \rho(\ell m), \rho(\ell m)] : \rho \in P_{\ell, m}\}$$

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Example: $\ell = 3, m = 2,$

$$\rho = [3, 1, 2, 2, 1, 3] \in P_{\ell, m},$$

$$\pi = [3, 3, 1, 1, 2, 2, 2, 2, 1, 1, 3, 3] \in DP_{\ell, m}$$

Lower Bound on $|A(n, k)|$

- Let $S(a, b)$ be the set of all permutations on the set $\{a, a + 1, \dots, a + b\}$.
- Let $\rho \in P_{\ell, m}$, and for $1 \leq i \leq \ell$, let $\gamma_i \in S(m(i - 1) + 1, m)$. Let $\alpha = \rho(\gamma_1, \gamma_2, \dots, \gamma_\ell) \in S_n, n = \ell m$, defined by $\alpha(j) = \gamma_i(r)$, where $\rho(j) = i$ is the r th copy of i in ρ .

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Example: If $\rho = [3, 1, 2, 2, 1, 3] \in P_{3, 2}, \gamma_1 = [1, 2], \gamma_2 = [4, 3], \gamma_3 = [6, 5]$, then $\rho(\gamma_1, \gamma_2, \gamma_3) = [6, 1, 4, 3, 2, 5]$.

Lower Bound on $|A(n, k)|$

Construction: Let $n = \ell k$, where k is even and let

$$C_{\ell, k} = \left\{ \pi(\gamma_1, \gamma_2, \dots, \gamma_\ell) : \begin{array}{l} \pi \in DP_{\ell, \frac{k}{2}} \\ \gamma_i \in S(k \cdot (i - 1) + 1, ki) \end{array} \right\}$$

Lower Bound on $|A(n, k)|$

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Example: $\ell = 3, k = 4,$

$$\pi = [3, 3, 1, 1, 2, 2, 2, 2, 1, 1, 3, 3] \in DP_{3, 2}, \gamma_1 = [1, 2, 4, 3],$$

$$\gamma_2 = [7, 5, 6, 8], \gamma_3 = [11, 12, 9, 10], \text{ then}$$

$$\pi(\gamma_1, \gamma_2, \gamma_3) = [11, 12, 1, 2, 7, 5, 6, 8, 4, 3, 9, 10] \in C_{3, k}.$$

Lower Bound on $|A(n, k)|$

- $C_{\ell, k} \subset A(n, k)$.
- $|C_{\ell, k}| = \frac{\frac{n!}{2!}}{\left(\frac{k!}{2!}\right)^\ell} \cdot (\ell!)^k$.

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$$C(\epsilon) \geq \frac{1 + \epsilon}{2}$$

Upper Bound on $|A(n, k)|$

- Let $\psi: A(n, k) \rightarrow \mathbb{Z}^n$

$$\psi(\sigma) = (\sigma(1), \sigma(2) - \sigma(1), \dots, \sigma(n) - \sigma(n-1)).$$

Upper Bound on $|A(n, k)|$

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Example: $\sigma = [6, 2, 1, 3, 4, 5] \in A(6, 1)$

$$\psi(\sigma) = (6, -4, -1, 2, 1, 1)$$

Upper Bound on $|A(n, k)|$

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- For $\mathbf{x} \in \psi(A(n, k)),$

$$|\{i : 2 \leq i \leq n, |x_i| \leq 2k\}| \geq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

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- For $J \subseteq \{2, 3, \dots, n\}$ let
$$B_J = \{\mathbf{x} \in \psi(A(n, k)) : |x_i| \leq 2k \text{ iff } i \in J\}.$$
- $|B_J| \leq (2k)^{|J|} (2n)^{n-1-|J|} n = 4^{n-1} k^{|J|} n^{n-|J|} \leq 4^n k^{\frac{n}{2}} n^{\frac{n}{2}}.$

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- $|\psi(A(n, k))| = \sum_{J \subseteq \{2, 3, \dots, n\}} |B_J| \leq 2^n 4^n k^{\frac{n}{2}} n^{\frac{n}{2}}.$

Capacity of Two-Neighbor Constrained Codes

$$\frac{\log \frac{\left(\frac{n}{2}\right)!}{\left(\frac{k}{2}\right)!^{\frac{n}{k}} \cdot \left(\frac{n}{k}\right)!^k}{\log n!}} \leq \frac{\log |A(n, k)|}{\log n!} \leq \frac{\log 2^n 4^n k^{\frac{n}{2}} n^{\frac{n}{2}}}{\log n!}$$

Capacity of Two-Neighbor Constrained Codes

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\downarrow
 $\frac{1 + \epsilon}{2}$

\downarrow
 $C(\epsilon)$

\downarrow
 $\frac{1 + \epsilon}{2}$

$$k = n^\epsilon, \quad 0 \leq \epsilon \leq 1$$

Capacity of Two-Neighbor Constraint Codes

$$\frac{\log \frac{\frac{n}{2}!}{\left(\frac{k}{2}!\right)^{\frac{n}{k}}}}{\log \frac{4^n k^{\frac{n}{2}} n^{\frac{n}{2}}}{n!}}$$
$$\frac{1 + \epsilon}{2} \qquad C(\epsilon) \qquad \frac{1 + \epsilon}{2}$$

$$k = n^\epsilon, \quad 0 \leq \epsilon \leq 1$$

Constrained Error-Correcting Codes

- For two permutations $\sigma, \pi \in S_n$ their **Kendall's τ -distance** is

$$d_K(\sigma, \pi) = \left| \left\{ (i, j) : \begin{array}{l} \sigma^{-1}(i) < \sigma^{-1}(j) \\ \pi^{-1}(i) > \pi^{-1}(j) \end{array} \right\} \right|$$

- Their **inversion distance** is

$$d_I(\sigma, \pi) = d_K(\sigma^{-1}, \pi^{-1}) = \left| \left\{ (i, j) : \begin{array}{l} \sigma(i) < \sigma(j) \\ \pi(i) > \pi(j) \end{array} \right\} \right|$$

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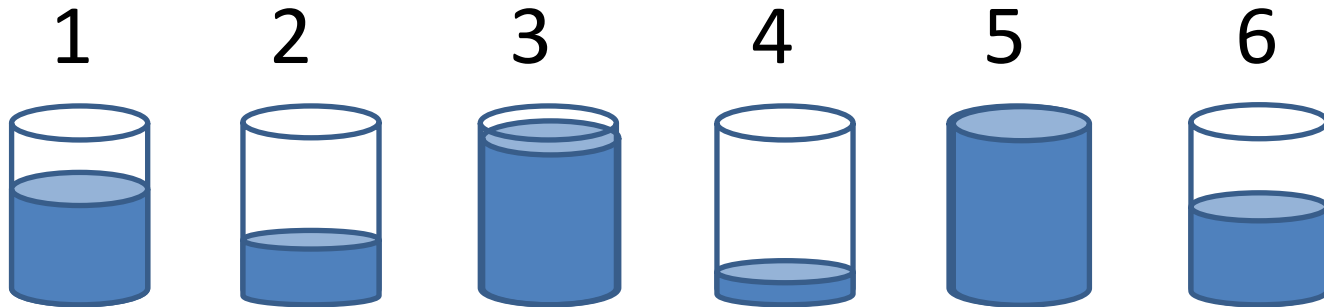
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Example: $\sigma = [3, 1, 2, 5, 4], \pi = [5, 1, 3, 4, 2]$

$$d_I(\sigma, \pi) = |\{(1, 4), (1, 5), (3, 5)\}| = 3$$

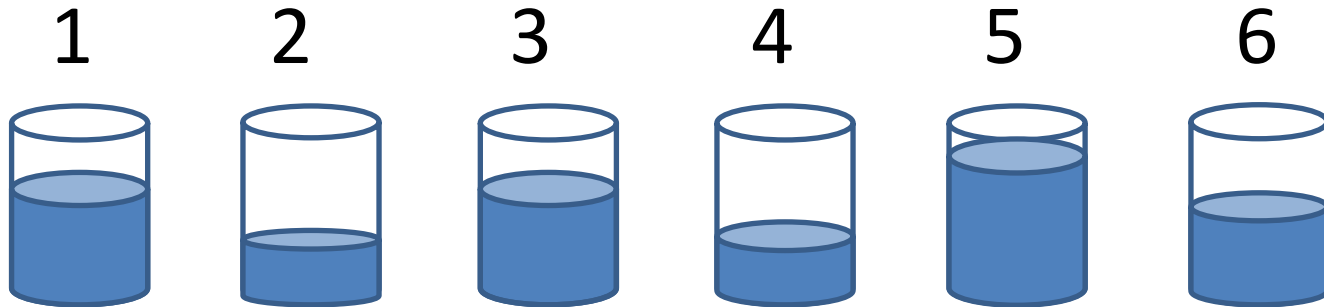
Error-Correction for Rank Modulation



$$\sigma = [4, 2, 5, 1, 6, 3]$$

$$\sigma^{-1} = [4, 2, 6, 1, 3, 5]$$

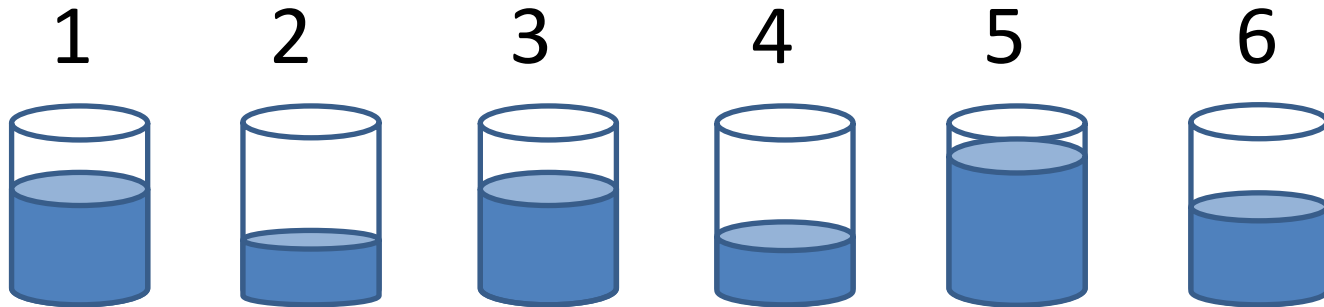
Error-Correction for Rank Modulation



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Error-Correction for Rank Modulation



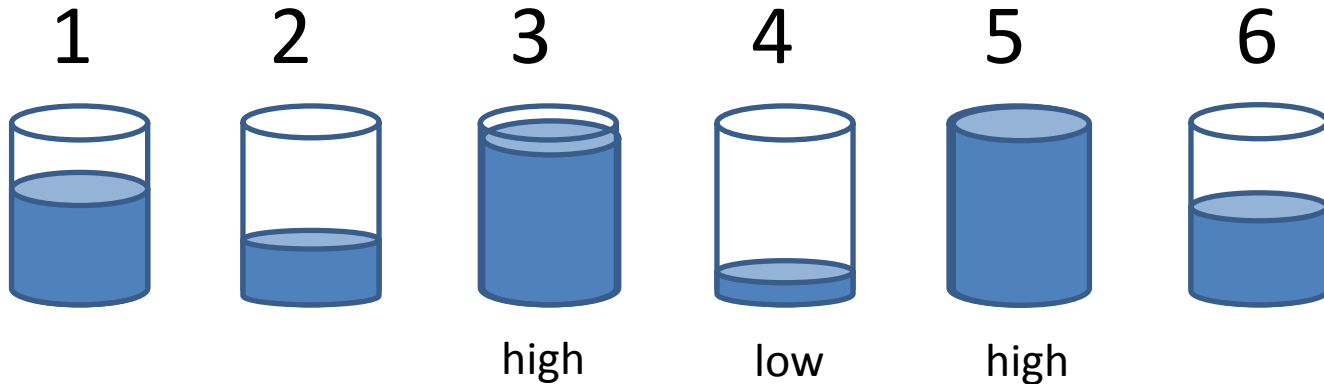
$$\sigma = [4, 1, 5, 2, 6, 3]$$

Inversion
metric

$$\sigma^{-1} = [2, 4, 6, 1, 3, 5]$$

Kendall's τ -
metric

Error-Correction for Rank Modulation



$$\sigma = [4, 2, 5, 1, 6, 3]$$

high low high

$$\sigma^{-1} = [4, 2, 6, 1, 3, 5]$$

Error-Correcting Codes

Barg & Mazumdar 10: The capacity of codes in S_n with minimum inversion distance d , where $d = n^\delta$, is

$$C^{EC}(\delta) = \begin{cases} 1 & 0 \leq \delta \leq 1 \\ 2 - \delta & 1 < \delta \leq 2 \end{cases}$$

Single-Neighbor-Constrained ECC

Sala & Dolecek 13: The capacity of single-neighbor k -constrained codes with minimum inversion distance d , where $k = n^\epsilon$, $0 \leq \epsilon \leq 1$, $d = n^\delta$, is

$$C^{SEC}(\epsilon, \delta) = \begin{cases} \epsilon & 0 \leq \delta \leq 1 \\ 1 + \epsilon - \delta & 1 < \delta < 1 + \epsilon \\ 0 & 1 + \epsilon \leq \delta \leq 2 \end{cases}$$

Two-Neighbor Constrained EEC

- Let $E(n, k, d)$ be the size of the largest two-neighbor k -constrained code with minimum inversion distance d .

$$C^{TEC}(\epsilon, \delta) = \lim_{n \rightarrow \infty} \frac{\log E(n, k, d)}{\log n!}, \quad \begin{array}{l} k = n^\epsilon \\ d = n^\delta \end{array}$$

Two-Neighbor Constrained EEC

Theorem:

$$C^{TEC}(\epsilon, \delta) = \begin{cases} \frac{1 + \epsilon}{2} & 0 \leq \delta \leq 1 \\ \frac{1 + \epsilon}{2} + 1 - \delta & 1 < \delta < 1 + \epsilon \\ 1 - \frac{\delta}{2} & 1 + \epsilon \leq \delta \leq 2 \end{cases}$$

Two-Neighbor Constrained EEC

- $B_I(n, r, \sigma) = \{\pi \in S_n : d_I(\pi, \sigma) \leq r\}$.
- $B_I(n, r)$ -the size of a ball of radius r in S_n .

$$\frac{|A(n, k)|}{B_I(n, d - 1)} \leq E(n, k, d) \leq |A(n, k)|$$

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$$\frac{|A(n, k)|}{B_I(n, d - 1)} \leq E(n, k, d) \leq |A(n, k)|$$

BaMa10,LoPr03,Mar01: Let $r = n^\delta$, then

$$B_I(n, r) \leq \begin{cases} c_1^n & 0 \leq \delta \leq 1 \\ c_2^n (n^{\delta-1})^n & 1 < \delta \leq 2 \end{cases}$$

Two-Neighbor Constrained EEC

- $B_I(n, r, \sigma) = \{\pi \in \mathcal{S} : d_I(\pi, \sigma) \leq r\}$

- $C^{TEC}(\epsilon, \delta) = \frac{1+\epsilon}{2}, \quad 0 \leq \delta \leq 1.$

$$C^{TEC}(\epsilon, \delta) \geq \frac{1+\epsilon}{2} + 1 - \delta, \quad 1 < \delta \leq 2.$$

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Two-Neighbor Constrained EEC

- Let $H_n = \{1, 2, \dots, n\}^n$.
- For $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in H_n$ their **Manhattan distance** is

$$d_M(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$$

Two-Neighbor Constrained EEC

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$$d_M(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$$

- For every $\sigma, \pi \in S_n$
$$\frac{1}{2} d_M(\sigma, \pi) \leq d_I(\sigma, \pi) \leq d_M(\sigma, \pi)$$

Two-Neighbor Constrained EEC

- $V(n, k) = \left\{ \mathbf{x} \in H_n : \begin{array}{l} \mathbf{x} \text{ satisfies the two} \\ \text{neighbor } k - \text{constraint} \end{array} \right\}$
- $A(n, k) \subset V(n, k) \subset H_n$.
- $B_M(V(n, k), \alpha, r) = \{ \beta \in V(n, k) : d_M(\alpha, \beta) \leq r \}$.

Two-Neighbor Constrained EEC

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$$\frac{|V(n, k)|}{\max_{\mathbf{x} \in V(n, k)} |B_M(V(n, k), \mathbf{x}, 2d - 1)|} \leq E(n, k, d)$$

$$\leq \frac{|V(n, k)|}{\min_{\mathbf{x} \in V(n, k)} \left| B_M \left(V(n, k), \mathbf{x}, \frac{d-1}{2} \right) \right|}$$

Two-Neighbor Constrained EEC

Lemma: Let $k = n^\epsilon$, $r = n^\delta$, $0 \leq \epsilon \leq 1$, then

$$|B_M(V(n, k), \mathbf{x}, r)| \geq \begin{cases} \left(\frac{n^{\delta-1}}{2}\right)^n & 1 < \delta \leq 2 \\ \left(\frac{n^{\delta-1+\epsilon}}{c}\right)^{\frac{n}{2}} & 1 + \epsilon \leq \delta \leq 2 \end{cases}$$

Two-Neighbor Constrained EEC

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$$C^{TEC}(\epsilon, \delta) \leq \begin{cases} \frac{1+\epsilon}{2} + 1 - \delta & 1 < \delta < 1 + \epsilon \\ 1 - \frac{\delta}{2} & 1 + \epsilon < \delta \leq 2 \end{cases}$$

Two-Neighbor Constrained EEC

Lemma: Let $k = n^\epsilon$, $r = n^\delta$, $0 \leq \epsilon \leq 1$, then

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Two-Neighbor Constrained EEC

Lemma: Let $k = n^\epsilon$, $r = n^\delta$, $0 \leq \epsilon \leq 1$, then

$$|B_M(V(n, k), \mathbf{x}, r)| \leq \left(\frac{n^{\delta-1+\epsilon}}{c} \right)^{\frac{n}{2}}, \quad 1 + \epsilon \leq \delta \leq 2$$

$$C^{TEC}(\epsilon, \delta) \geq 1 - \frac{\delta}{2}, \quad 1 + \epsilon < \delta \leq 2$$

Proof of the Lemma

- Let $m = \left\lfloor \frac{n}{2} \right\rfloor$ and let ϕ be an injection that maps $\mathbf{y} \in B_M(V(n, k), \mathbf{x}, r)$ into $(\mathbf{u}, \mathbf{a}, \mathbf{v}, \mathbf{b})$,
 $\mathbf{u} \in \{0, 1, \dots, n\}^m$, where $\sum_{i=1}^m u_i \leq r$, $\mathbf{a} \in \{0, 1\}^m$,
 $\mathbf{v} \in \{0, 1, \dots, k\}^{n-m}$, $\mathbf{b} \in \{0, 1, 2, 3\}^{n-m}$.

Proof of the Lemma

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 $\mathbf{u} \in \{0, 1, \dots, n\}^m$, where $\sum_{i=1}^m u_i \leq r$, $\mathbf{a} \in \{0, 1\}^m$,
 $\mathbf{v} \in \{0, 1, \dots, k\}^{n-m}$, $\mathbf{b} \in \{0, 1, 2, 3\}^{n-m}$.
- $|B_M(V(n, k), \mathbf{x}, r)| \leq \binom{r + m}{m} (8k)^{\frac{n}{2}}$
- $\binom{d + m}{m} \leq c_1^n (n^{\delta-1})^{\frac{n}{2}}$.

Proof of the Lemma

- Let $m = \left\lceil \frac{n}{2} \right\rceil$ and let ϕ be an injection that maps $\mathbf{y} \in B_M(V(n, k), \mathbf{x}, r)$ into $(\mathbf{u}, \mathbf{a}, \mathbf{v}, \mathbf{b})$, $\mathbf{u} \in (\{0, 1, \dots, n\})^m$, where $\sum_{i=1}^m u_i \leq r$, $\mathbf{a} \in \{0, 1\}^m$, $\mathbf{v} \in \{0, 1, \dots, k\}^{n-m}$, $\mathbf{b} \in \{0, 1, 2, 3\}^{n-m}$.
- $|B_M(V(n, k), \mathbf{x}, d)| \leq \binom{d+m}{m} (4k)^{\frac{n}{2}}$
- $\binom{d+m}{m} \leq c_1^n (n^{\delta-1})^{\frac{n}{2}}$.

$$|B_M(V(n, k), \mathbf{x}, d)| \leq c^n (n^{\delta-1+\epsilon})^{\frac{n}{2}}$$

Proof of the Lemma

- For $\mathbf{y} \in B_M(V(n, k), \mathbf{x}, r)$, $\phi(\mathbf{y}) = (\mathbf{u}, \mathbf{a}, \mathbf{v}, \mathbf{b})$,

$$(u_i, a_i) = \begin{cases} (y_{2i-1} - x_{2i-1}, 0) & 0 \leq y_{2i-1} - x_{2i-1} \\ (x_{2i-1} - y_{2i-1}, 1) & \textit{otherwise} \end{cases}$$

$$(v_i, b_i) = \begin{cases} (y_{2i} - y_{2i-1}, 0) & 0 \leq y_{2i} - y_{2i-1} \leq k \\ (y_{2i-1} - y_{2i}, 1) & 0 < y_{2i-1} - y_{2i} \leq k \\ (y_{2i+1} - y_{2i}, 2) & |y_{2i} - y_{2i-1}| > k, 0 \leq y_{2i+1} - y_{2i} \leq k \\ (y_{2i} - y_{2i+1}, 3) & |y_{2i} - y_{2i-1}| > k, 0 < y_{2i} - y_{2i+1} \leq k \end{cases}$$

Proof of the Lemma

- $d_M(\mathbf{y}, \mathbf{x}) \leq r \Rightarrow \sum_{i=1}^m u_i \leq r.$
- $\mathbf{y} \in V(n, k) \Rightarrow \mathbf{v} \in \{0, 1, 2, \dots, k\}^{n-m}.$
- ϕ is an inversion.

Asymmetric Two-Neighbor Constraint

- $\sigma \in S_n$ satisfies the **asymmetric two-neighbor k -constraint** if for every $2 \leq i \leq n - 1$, either $\sigma(i - 1) - \sigma(i) \leq k$ or $\sigma(i + 1) - \sigma(i) \leq k$.

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Example:

$[3, 6, 1, 2, 5, 4]$ satisfies the asymmetric two-neighbor 1-constraint.

Asymmetric Two-Neighbor Constraint

- Let $A^{asym}(n, k) \subset S_n$ consists of all permutations that satisfy the asymmetric two-neighbor k -constraint.
- Let $C^{asym}(\epsilon) = \lim_{n \rightarrow \infty} \frac{\log |A^{asym}(n, k)|}{\log n!}$,
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Theorem: $C^{asym}(\epsilon) = 1$.

Lower Bound on $|A^{asym}(n, k)|$

- For a set I , let I^{\nearrow} be the increasing ordering of I , and let I^{\searrow} be the decreasing ordering of I .

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Example $I = \{10, 2, 8, 4, 3, 9\}$.

$$I^{\nearrow} = 2, 3, 4, 8, 9, 10$$

$$I^{\searrow} = 10, 9, 8, 4, 3, 2$$

Lower Bound on $|A^{asym}(n, k)|$

- Let $\pi \in C_{r,2}$.

$$\pi = \rho(\gamma_1, \gamma_2, \dots, \gamma_r), \quad \gamma_i \in S(2i-1, 2i)$$

$$\rho = [\sigma(1), \sigma(1), \sigma(2), \sigma(2), \dots, \sigma(r), \sigma(r)], \quad \sigma \in S_r$$

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Example: $r = 2$, $\sigma = [3, 1, 2]$,

$$\gamma_1 = [2, 1], \gamma_2 = [3, 4], \gamma_3 = [6, 5].$$

$$\rho = [3, 3, 1, 1, 2, 2]$$

$$\pi = [6, 5, 2, 1, 3, 4].$$

Lower Bound on $|A^{asym}(n, k)|$

- Let $r \leq \frac{n-1}{4}$ and let $C_r \subset S_n$ s.t. $\sigma \in C_r$ if there exist:
 - 1) A partition of $\{2r + 1, 2r + 2, \dots, n\}$ into $2r + 2$ classes $I_1, I_2, \dots, I_{2r+2}$ (nonempty, except I_{2r+2}).
 - 2) A permutation $\pi \in C_{r,2}$.

and $\sigma = [I_1^{\nearrow}, I_2^{\searrow}, \pi(1), \pi(2), I_3^{\nearrow}, I_4^{\searrow}, \pi(3), \pi(4), \dots, \pi(2r-1), \pi(r), I_{2r+1}^{\nearrow}, I_{2r+2}^{\searrow}]$

Lower Bound on $|A^{asym}(n, k)|$

Example: $n = 15, r = 2,$

$$I_1 = \{5, 8, 10\}, \quad I_2 = \{6, 12\},$$

$$I_3 = \{7\}, \quad I_4 = \{9, 15\},$$

$$I_5 = \{13\}, \quad I_6 = \{11, 14\}$$

$$\pi = [4, 3, 1, 2] \in C_{2,2}$$

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Lower Bound on $|A^{asym}(n, k)|$

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Construction: Let

$$C_n^{asym} = \bigcup_{r=0}^{\frac{n-1}{4}} C_r,$$

- $C_n^{asym} \subset A^{asym}(n, 1)$ and

$$|C_n^{asym}| \geq \sum_{r=0}^{\frac{n-1}{4}} \frac{1}{2} (2r)! S(n - 2r, 2r)$$

Capacity of Asymmetric Two-Neighbor Constraint Codes

- Let $r = \lceil \delta n \rceil$, $0 < \delta < \frac{1}{4}$.

$$\begin{aligned} \frac{\log |A^{asym}(n, 1)|}{\log n!} &\geq \frac{\log |C_n^{asym}|}{\log n!} \geq \frac{\log |C_r|}{\log n!} \\ &\geq \frac{\log \frac{1}{2} (2r)! S(n - 2r, 2r)}{\log n!} \geq 1 - \delta. \end{aligned}$$

Capacity of Asymmetric Two-Near Neighbor Constraint Codes

- Let $r = \lceil \delta n \rceil$, $0 < \delta < \frac{1}{4}$.
- $\frac{\log |A^{asym}(n, 1)|}{n} \geq \frac{\log |C_n^{asym}|}{n}$

$$C^{asym}(\epsilon) \geq C^{asym}(0) = 1$$

Conclusion

- Capacity of two-neighbor constrained codes.
- Capacity of two-neighbor constrained error-correcting codes, with the inversion metric.
- Capacity of asymmetric two-neighbor constrained codes.

Thank You!