Constrained Codes for Rank Modulation

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Coding Theory Seminar

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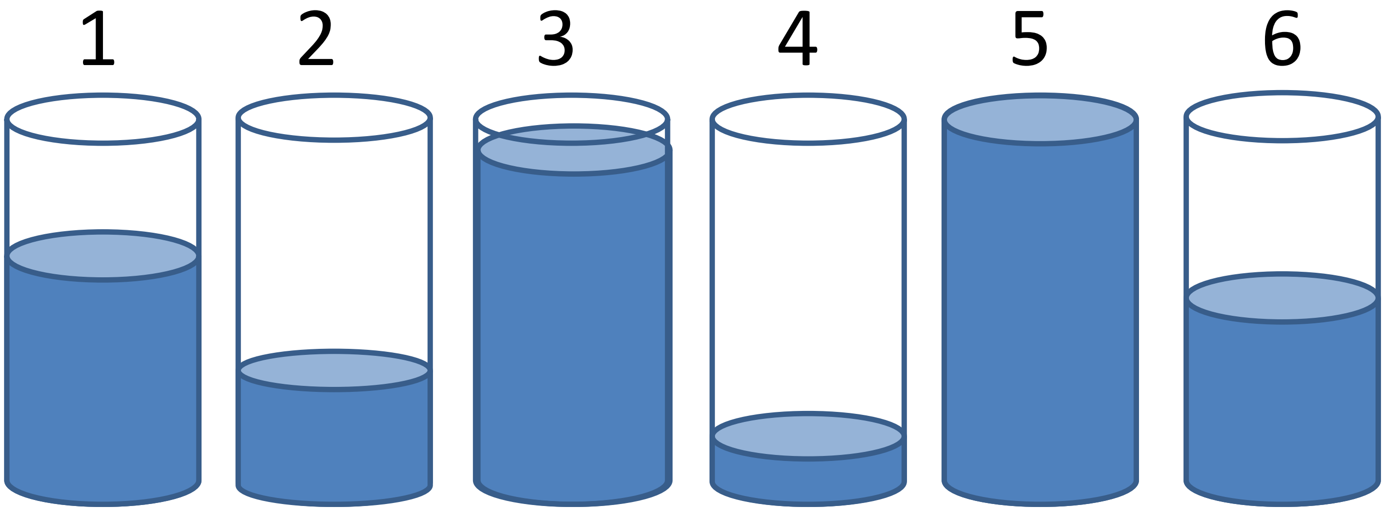
Outline

- Introduction
- Two-neighbor constraint.
- Constrained error-correcting codes.
- Asymmetric two-neighbor constraint.

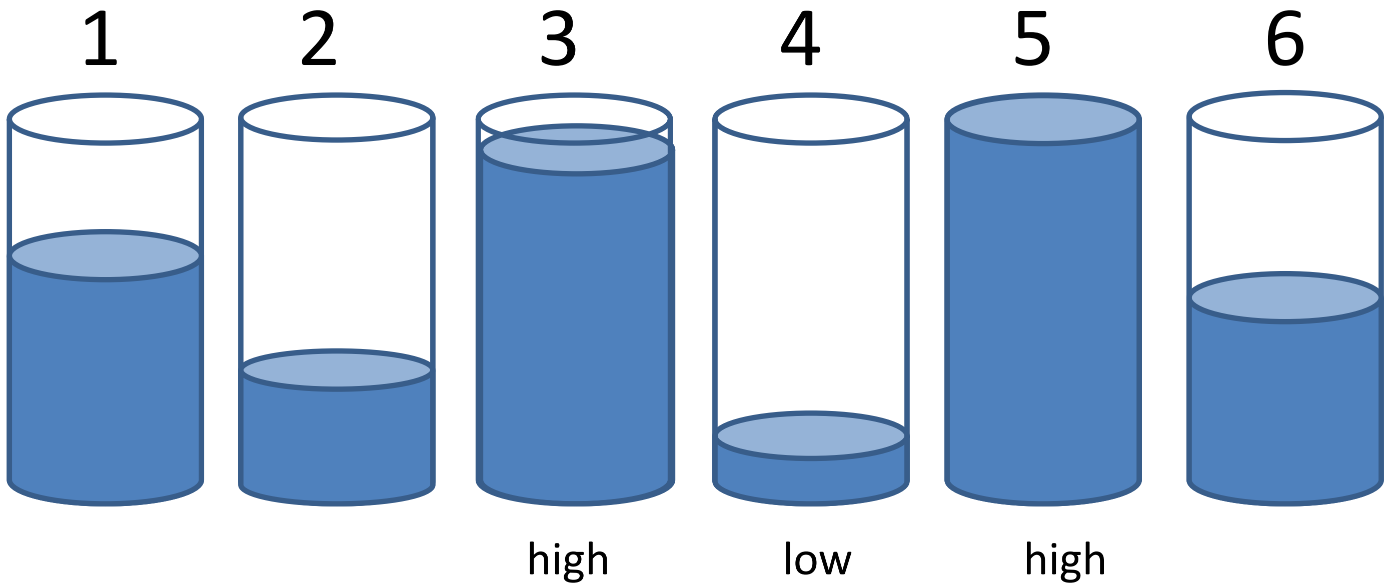
Rank Modulation for Flash Memory



[4,2,5,1,6,3]

Codewords are permutations!

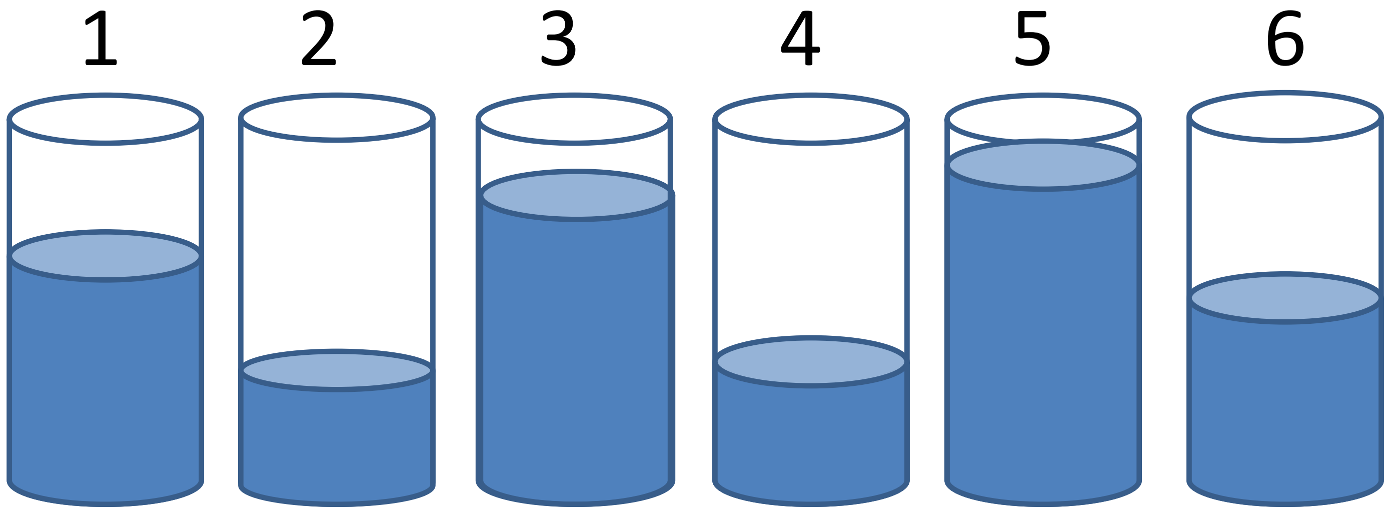
Inter-Cell Interference



[4,2,5,1,6,3]

high low high

Inter-Cell Interference



[4,**1**,5,**2**,6,3]

Single-Neighbor-Constraint

- S_n-the set of all permutations on n-elements.

- $\sigma \in S_n$ satisfies the **single-neighbor *k*-constraint** if for every $1 \leq i \leq n-1$, $|\sigma(i) - \sigma(i+1)| \leq k$.

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Example:

[5,6,4,2,1,3] satisfies the single-neighbor 2-constraint.

Single-Neighbor Constraint

- Let $A^{SC}(n,k) \subset S_n$ consists of all permutations that satisfy the single-neighbor k-constraint.
- Let $C^{SC}(\epsilon) = \lim_{n\to\infty} \frac{\log |A^{SC}(n,k)|}{\log n!}, \quad k = n^{\epsilon}$.

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Sala & Dolecek 13: $C^{SC}(\epsilon) = \epsilon$, for all $0 \le \epsilon \le 1$.

Two-Neighbor Constraint

- $\sigma \in S_n$ satisfies the **two-neighbor $\boldsymbol{k}$-constraint** if for every $2 \leq i \leq n-1$, either $|\sigma(i) - \sigma(i-1)| \leq k$ or $|\sigma(i) - \sigma(i+1)| \leq k$.

Two-Neighbor Constraint

- $\sigma \in S_n$ satisfies the **two-neighbor k-constraint** if for every $2 \leq i \leq n-1$, either $|\sigma(i) - \sigma(i-1)| \leq k$ or $|\sigma(i) - \sigma(i+1)| \leq k$.

Example:

[6,1,2,5,4,3] satisfies the two-neighbor 1-constraint.

Two-Neighbor Constraint

- Let $A(n,k) \subset S_n$ consists of all permutations that satisfy the two-neighbor k-constraint.
- Let $C(\epsilon) = \lim_{n\to\infty} \frac{\log |A(n,k)|}{\log n!}$, $\quad k = n^{\epsilon}$.

Two-Neighbor Constraint

- Let $A(n,k) \subset S_n$ consists of all permutations that satisfy the two-neighbor k-constraint.
- Let $C(\epsilon) = \lim_{n\to\infty} \frac{\log |A(n,k)|}{\log n!}, \qquad k = n^{\epsilon}$.

Theorem: For all $0 \leq \epsilon \leq 1$, $C(\epsilon) = \frac{1+\epsilon}{2}$.

Lower Bound on $|A(n,k)|$

- Let $P_{\ell,m}$ be the set of all **multi-permutations** on

$$\{\underbrace{1,1,\ldots,1}_{m\text{ times}},\underbrace{2,2,\ldots,2}_{m\text{ times}},\ldots,\underbrace{\ell,\ell,\ldots,\ell}_{m\text{ times}}\}.$$

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Example: $\ell = 3, m = 2,$

$$\rho = [3,1,2,2,1,3] \in P_{\ell,m}.$$

Lower Bound on $|A(n, k)|$

- Let $DP_{\ell,m}$ be the set

$$\{[\rho(1), \rho(1), \rho(2), \rho(2), \ldots, \rho(\ell m), \rho(\ell m)] : \rho \in P_{\ell,m}\}$$

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$$\rho = [3,1,2,2,1,3] \in P_{\ell,m},$$

$$\pi = [3,3,1,1,2,2,2,2,1,1,3,3] \in DP_{\ell,m}$$

Lower Bound on $|A(n,k)|$

- Let $S(a,b)$ be the set of all permutations on the set $\{a, a+1, \dots, a+b\}$.
- Let $\rho \in P_{\ell,m,}$ and for $1 \leq i \leq \ell$, let $\gamma_i \in S(m(i-1)+1, m)$. Let $\alpha = \rho(\gamma_1, \gamma_2, \dots, \gamma_\ell) \in S_n, n = \ell m$, defined by $\alpha(j) = \gamma_i(r)$, where $\rho(j) = i$ is the rth copy of i in ρ.

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Example: If $\rho = [3,1,2,2,1,3] \in P_{3,2}, \gamma_1 = [1,2]$,

$\gamma_2 = [4,3], \gamma_3 = [6,5]$, then

$\rho(\gamma_1, \gamma_2, \gamma_3) = [6,1,4,3,2,5]$.

Lower Bound on $|A(n,k)|$

Construction: Let $n = \ell k$, where k is even and let

$$C_{\ell,k} = \left\{ \pi(\gamma_1, \gamma_2, \ldots, \gamma_\ell): \begin{array}{c} \pi \in DP_{\ell,\frac{k}{2}} \\ \gamma_i \in S(k \cdot (i-1) + 1, ki) \end{array} \right\}$$

Lower Bound on $|A(n,k)|$

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Example: $\ell = 3, k = 4,$

$\pi = [3,3,1,1,2,2,2,2,1,1,3,3] \in DP_{3,2}, \gamma_1 = [1,2,4,3],$

$\gamma_2 = [7,5,6,8], \gamma_3 = [11,12,9,10]$, then

$\pi(\gamma_1, \gamma_2, \gamma_3) = [11,12,1,2,7,5,6,8,4,3,9,10] \in C_{3,k}.$

Lower Bound on $|A(n,k)|$

- $C_{\ell,k} \subset A(n,k)$.
- $\left|C_{\ell,k}\right| = \dfrac{\frac{n}{2}!}{\left(\frac{k}{2}!\right)^{\ell}} \cdot (\ell!)^{k}$.

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$$C(\epsilon) \geq \frac{1+\epsilon}{2}$$

Upper Bound on $|A(n,k)|$

- Let $\psi: A(n,k) \to \mathbb{Z}^n$

$$\psi(\sigma) = \big(\sigma(1), \sigma(2) - \sigma(1), \dots, \sigma(n) - \sigma(n-1)\big).$$

Upper Bound on $|\boldsymbol{A(n,k)}|$

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Example: $\sigma = [6,2,1,3,4,5\,] \in A(6,1)$

$$\psi(\sigma) = (6, -4, -1,2,1,1)$$

Upper Bound on $|\boldsymbol{A}(\boldsymbol{n},\boldsymbol{k})|$

- $|A(n,k)| = \left|\psi\left(A(n,k)\right)\right| = |\{\psi(\sigma): \sigma \in A(n,k)\}|$

Upper Bound on $|A(n,k)|$

- $|A(n,k)| = \left|\psi\big(A(n,k)\big)\right| = |\{\psi(\sigma) : \sigma \in A(n,k)\}|$.
- For $\boldsymbol{x} \in \psi\big(A(n,k)\big)$,

$$|\{i : 2 \le i \le n, |x_i| \le 2k\}| \ge \left\lfloor \frac{n-1}{2} \right\rfloor.$$

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- For $J \subseteq \{2,3,\dots,n\}$ let
$$B_J = \left\{\boldsymbol{x} \in \psi\big(A(n,k)\big)\colon\ |x_i| \le 2k\ iff\ i \in J\right\}.$$
- $\left|B_J\right| \le (2k)^{|J|}(2n)^{n-1-|J|}n = 4^{n-1}k^{|J|}n^{n-|J|} \le 4^n k^{\frac{n}{2}} n^{\frac{n}{2}}$.

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- $\left|\psi\left(A(n,k)\right)\right| = \sum_{J \subseteq \{2,3,\dots,n\}} \left|B_J\right| \leq 2^n 4^n k^{\frac{n}{2}} n^{\frac{n}{2}}$.

Capacity of Two-Neighbor Constrained Codes

$$\frac{\log \frac{\frac{n}{2}!}{\left(\frac{k}{2}!\right)^{\frac{n}{k}}} \cdot \left(\frac{n}{k}!\right)^{k}.}{\log n!} \le \frac{\log |A(n,k)|}{\log n!} \le \frac{\log 2^{n} 4^{n} k^{\frac{n}{2}} n^{\frac{n}{2}}}{\log n!}$$

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$$\frac{1+\epsilon}{2} \qquad\qquad C(\epsilon) \qquad\qquad \frac{1+\epsilon}{2}$$

$k = n^{\epsilon}, \qquad 0 \leq \epsilon \leq 1$

Capacity of Two-Neighbor Constraint Codes

$$\frac{\log \frac{\frac{n}{2}!}{\left(\frac{k}{2}!\right)^{\frac{n}{k}}}}{\log \ldots}$$

$$\frac{\ldots 4^n k^{\frac{n}{2}} n^{\frac{n}{2}}}{\ldots \log n!}$$

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Constrained Error-Correcting Codes

- For two permutations $\sigma, \pi \in S_n$ their **Kendall's τ-distance** is

$$d_K(\sigma,\pi) = \left|\left\{(i,j) : \begin{matrix} \sigma^{-1}(i) < \sigma^{-1}(j) \\ \pi^{-1}(i) > \pi^{-1}(j) \end{matrix}\right\}\right|$$

- Their **inversion distance** is

$$d_I(\sigma,\pi) = d_K(\sigma^{-1},\pi^{-1}) = \left|\left\{(i,j) : \begin{matrix} \sigma(i) < \sigma(j) \\ \pi(i) > \pi(j) \end{matrix}\right\}\right|$$

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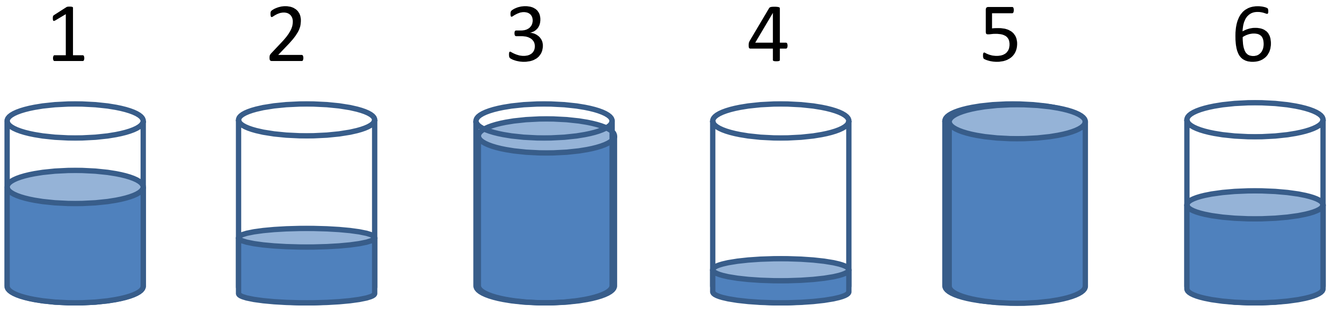
Example: $\sigma = [3,1,2,5,4], \pi = [5,1,3,4,2]$

$$d_I(\sigma, \pi) = |\{(1,4), (1,5), (3,5)\}| = 3$$

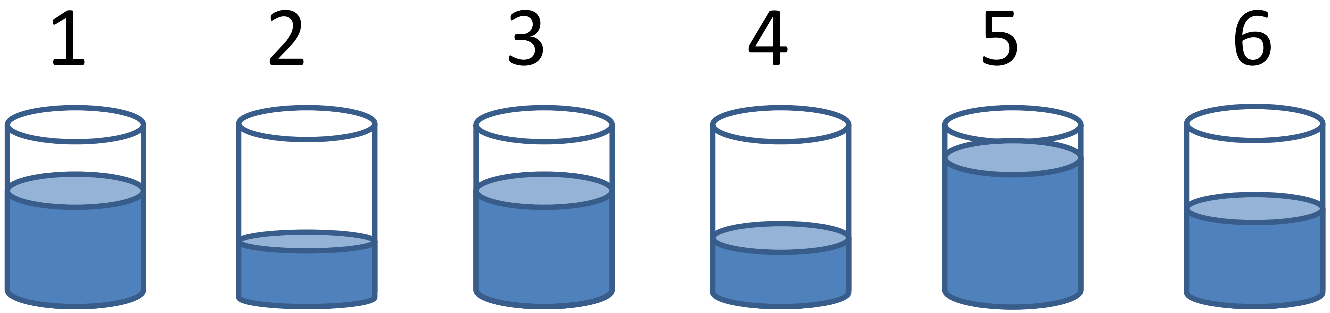
Error-Correction for Rank Modulation



$$\sigma = [4,2,5,1,6,3]$$

$$\sigma^{-1} = [4,2,6,1,3,5]$$

Error-Correction for Rank Modulation



$$\sigma = [4,\mathbf{1},5,\mathbf{2},6,3]$$

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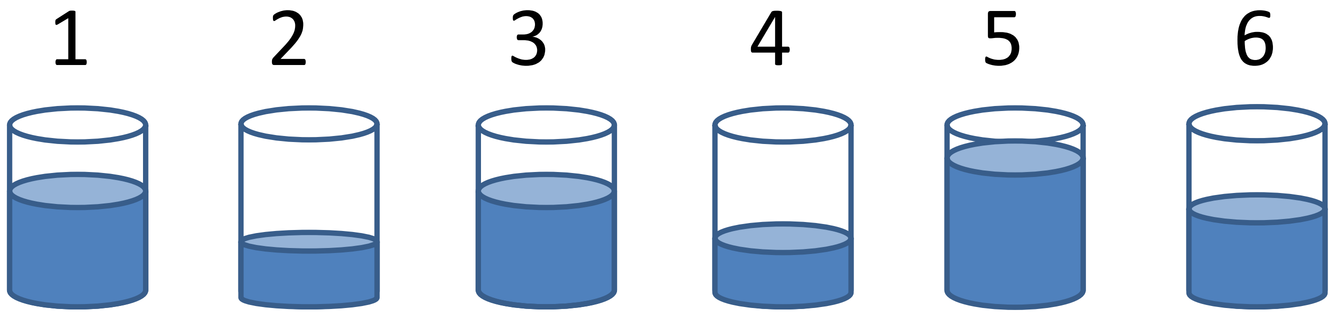
Error-Correction for Rank Modulation



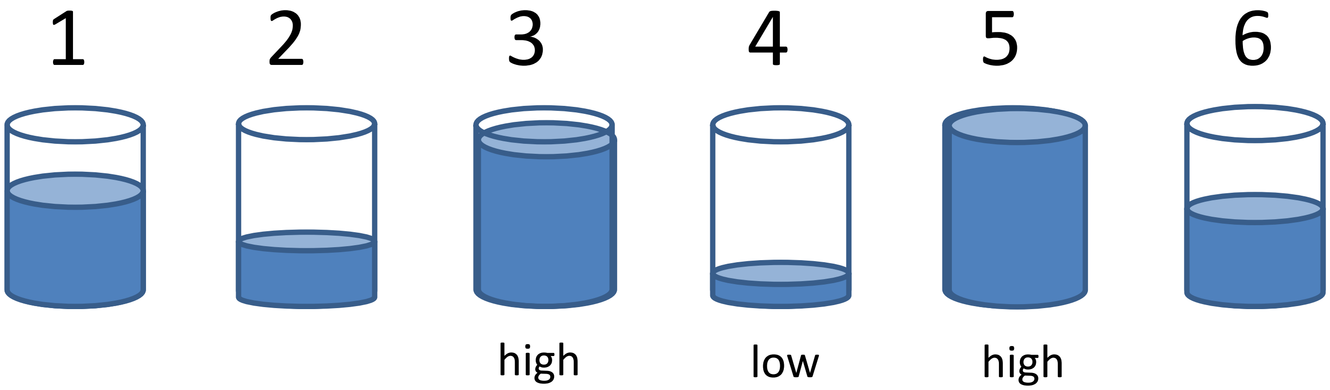
$$\sigma = [4,\mathbf{1},5,\mathbf{2},6,3]$$

Inversion metric

$$\sigma^{-1} = [\mathbf{2},\mathbf{4},6,1,3,5]$$

Kendall's τ-metric

Error-Correction for Rank Modulation



$$\sigma = [4,2,5,1,6,3]$$

high low high

$$\sigma^{-1} = [4,2,6,1,3,5]$$

Error-Correcting Codes

Barg & Mazumdar 10: The capacity of codes in S_n with minimum inversion distance d, where $d = n^{\delta}$, is

$$C^{EC}(\delta) = \begin{cases} 1 & 0 \leq \delta \leq 1 \\ 2 - \delta & 1 < \delta \leq 2 \end{cases}$$

Single-Neighbor-Constrained ECC

Sala & Dolecek 13: The capacity of single-neighbor k-constrained codes with minimum inversion distance d, where $k = n^{\epsilon},\ 0 \le \epsilon \le 1$, $d = n^{\delta}$, is

$$C^{SEC}(\epsilon,\delta) = \begin{cases} \epsilon & 0 \le \delta \le 1 \\ 1+\epsilon-\delta & 1 < \delta < 1+\epsilon \\ 0 & 1+\epsilon \le \delta \le 2 \end{cases}$$

Two-Neighbor Constrained EEC

- Let $E(n,k,d)$ be the size of the largest two-neighbor k-constrained code with minimum inversion distance d.

$$C^{TEC}(\epsilon,\delta) = \lim_{n\to\infty} \frac{\log E(n,k,d)}{\log n!}, \qquad \begin{array}{l} k = n^{\epsilon} \\ d = n^{\delta} \end{array}$$

Two-Neighbor Constrained EEC

Theorem:

$$C^{TEC}(\epsilon,\delta) = \begin{cases} \dfrac{1+\epsilon}{2} & 0 \le \delta \le 1 \\ \dfrac{1+\epsilon}{2} + 1 - \delta & 1 < \delta < 1+\epsilon \\ 1 - \dfrac{\delta}{2} & 1+\epsilon \le \delta \le 2 \end{cases}$$

Two-Neighbor Constrained EEC

- $B_I(n, r, \sigma) = \{\pi \in S_n : d_I(\pi, \sigma) \leq r\}$.
- $B_I(n, r)$-the size of a ball of radius r in S_n.

$$\frac{|A(n,k)|}{B_I(n, d-1)} \leq E(n,k,d) \leq |A(n,k)|$$

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$$\frac{|A(n,k)|}{B_I(n,d-1)} \leq E(n,k,d) \leq |A(n,k)|$$

BaMa10,LoPr03,Mar01: Let $r = n^{\delta}$, then

$$B_I(n,r) \leq \begin{cases} {c_1}^n & 0 \leq \delta \leq 1 \\ {c_2}^n\left(n^{\delta-1}\right)^n & 1 < \delta \leq 2 \end{cases}$$

Two-Neighbor Constrained EEC

- $B_I(n,r,\sigma) = \{\pi \in S_n : d_I(\pi,\sigma) \le r\}$
- B

$$C^{TEC}(\epsilon,\delta) = \frac{1+\epsilon}{2}, \quad 0 \le \delta \le 1.$$

$$C^{TEC}(\epsilon,\delta) \ge \frac{1+\epsilon}{2} + 1 - \delta, \quad 1 < \delta \le 2.$$

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Two-Neighbor Constrained EEC

- Let $H_n = \{1,2,\ldots,n\}^n$.
- For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n), \boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in H_n$ their **Manhattan distance** is

$$d_M(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

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$$d_M(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

- For every $\sigma, \pi \in S_n$

$$\frac{1}{2} d_M(\sigma, \pi) \le d_I(\sigma, \pi) \le d_M(\sigma, \pi)$$

Two-Neighbor Constrained EEC

- $V(n,k) = \left\{ \boldsymbol{x} \in H_n : \begin{matrix} \boldsymbol{x}\ satisfies\ the\ two \\ neighbor\ k - contstraint \end{matrix} \right\}$
- $A(n,k) \subset V(n,k) \subset H_n$.
- $B_M(V(n,k),\alpha,r) = \{\beta \in V(n,k) : d_M(\alpha,\beta\ ) \leq r\}$.

Two-Neighbor Constrained EEC

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- $A(n,k) \subset V(n,k) \subset H_n$.
- $B_M(V(n,k), \alpha, r) = \{\beta \in V(n,k) : d_M(\alpha, \beta\ ) \leq r\}$.

$$\frac{|V(n,k)|}{\max_{\boldsymbol{x} \in V(n,k)} |B_M(V(n,k), \boldsymbol{x}, 2d-1)|} \leq E(n,k,d)$$
$$\leq \frac{|V(n,k)|}{\min_{\boldsymbol{x} \in V(n,k)} \left|B_M\left(V(n,k), \boldsymbol{x}, \frac{d-1}{2}\right)\right|}$$

Two-Neighbor Constrained EEC

Lemma: Let $k = n^{\epsilon}, r = n^{\delta}, 0 \le \epsilon \le 1$, then

$$|B_M(V(n,k),\boldsymbol{x},r)| \ge \begin{cases} \left(\frac{n^{\delta-1}}{2}\right)^{n} & 1 < \delta \le 2 \\ \left(\frac{n^{\delta-1+\epsilon}}{c}\right)^{\frac{n}{2}} & 1+\epsilon \ \le \delta \le 2 \end{cases}$$

Two-Neighbor Constrained EEC

Lemma: Let $k = n^{\epsilon}, r = n^{\delta}, 0 \leq \epsilon \leq 1,$ then

$$|B_M(V(n,k),\boldsymbol{x},r)| \geq \begin{cases} \left(\frac{n^{\delta-1}}{2}\right)^{n} & 1 < \delta \leq 2 \\ \left(\frac{n^{\delta-1+\epsilon}}{c}\right)^{\frac{n}{2}} & 1+\epsilon \leq \delta \leq 2 \end{cases}$$

$$C^{TEC}(\epsilon,\delta) \leq \begin{cases} \frac{1+\epsilon}{2} + 1 - \delta & 1 < \delta < 1+\epsilon \\ 1 - \frac{\delta}{2} & 1+\epsilon < \delta \leq 2 \end{cases}$$

Two-Neighbor Constrained EEC

Lemma: Let $k = n^{\epsilon}, r = n^{\delta}, 0 \leq \epsilon \leq 1$, then

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Two-Neighbor Constrained EEC

Lemma: Let $k = n^{\epsilon}, r = n^{\delta}, 0 \le \epsilon \le 1,$ then

$$|B_M(V(n,k),\boldsymbol{x},r)| \le \left(\frac{n^{\delta-1+\epsilon}}{c}\right)^{\frac{n}{2}}, 1+\epsilon \le \delta \le 2$$

$$C^{TEC}(\epsilon,\delta) \ge 1 - \frac{\delta}{2}, \qquad 1+\epsilon < \delta \le 2$$

Proof of the Lemma

- Let $m = \left\lceil \frac{n}{2} \right\rceil$ and let ϕ be an injection that maps $\mathbf{y} \in B_M(V(n,k), \boldsymbol{x}, r)$ into $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{b})$, $\boldsymbol{u} \in \{0,1,\dots,n\}^m$, where $\sum_{i=1}^{m} u_i \le r$, $\boldsymbol{a} \in \{0,1\}^m$, $\boldsymbol{v} \in \{\mathbf{0},\mathbf{1},\dots,\boldsymbol{k}\}^{\boldsymbol{n-m}}$, $\boldsymbol{b} \in \{0,1,2,3\}^{\boldsymbol{n-m}}$.

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- $|B_M(V(n,k), \boldsymbol{x}, r)| \le \binom{r+m}{m} (8k)^{\frac{n}{2}}$
- $\binom{d+m}{m} \le c_1^n \left(n^{\delta-1}\right)^{\frac{n}{2}}$.

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- Let $m = \left\lceil \frac{n}{2} \right\rceil$ and let ϕ be an injection that maps $\mathbf{y} \in B_M(V(n,k), \boldsymbol{x}, r)$ into $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{b})$, $\boldsymbol{u} \in (\{0,1,\dots,n\})^m$, where $\sum_{i=1}^{m} u_i \le r, \boldsymbol{a} \in \{0,1\}^m$, $\boldsymbol{v} \in \{\mathbf{0}, \mathbf{1}, \dots, \boldsymbol{k}\}^{\boldsymbol{n}-\boldsymbol{m}}$, $\boldsymbol{b} \in \{0,1,2,3\}^{\boldsymbol{n}-\boldsymbol{m}}$.
- $|B_M(V(n,k), \boldsymbol{x}, d)| \le \binom{d+m}{m} (4k)^{\frac{n}{2}}$
- $\binom{d+m}{m} \le c_1^n \left(n^{\delta-1}\right)^{\frac{n}{2}}$.

$$|B_M(V(n,k), \boldsymbol{x}, d)| \le c^n \left(n^{\delta-1+\epsilon}\right)^{\frac{n}{2}}$$

Proof of the Lemma

- For $\mathbf{y} \in B_M(V(n,k),\boldsymbol{x},r)$, $\phi(\boldsymbol{y}) = (\boldsymbol{u},\boldsymbol{a},\boldsymbol{v},\boldsymbol{b})$,

$$(u_i,a_i) = \begin{cases} (y_{2i-1} - x_{2i-1}, 0) & 0 \leq y_{2i-1} - x_{2i-1} \\ (x_{2i-1} - y_{2i-1}, 1) & otherwise \end{cases}$$

$$(v_i,b_i) = \begin{cases} (y_{2i} - y_{2i-1}, 0) & 0 \leq y_{2i} - y_{2i-1} \leq k \\ (y_{2i-1} - y_{2i}, 1) & 0 < y_{2i-1} - y_{2i} \leq k \\ (y_{2i+1} - y_{2i}, 2) & |y_{2i} - y_{2i-1}| > k, 0 \leq y_{2i+1} - y_{2i} \leq k \\ (y_{2i} - y_{2i+1}, 3) & |y_{2i} - y_{2i-1}| > k, 0 < y_{2i} - y_{2i+1} \leq k \end{cases}$$

Proof of the Lemma

- $\mathrm{d}_M(\mathbf{y}, \mathbf{x}) \le r \Rightarrow \sum_{i=1}^{m} u_i \le r$.
- $\mathbf{y} \in V(n, k) \Rightarrow \boldsymbol{v} \in \{0,1,2, \dots, k\}^{n-m}$.
- ϕ is an inversion.

Asymmetric Two-Neighbor Constraint

- $\sigma \in S_n$ satisfies the **asymmetric two-neighbor $\boldsymbol{k}$-constraint** if for every $2 \leq i \leq n-1$, either $\sigma(i-1)-\sigma(i) \leq k$ or $\sigma(i+1)-\sigma(i) \leq k$.

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Example:

$[3,6,1,2,5,4]$ satisfies the asymmetric two-neighbor 1-constraint.

Asymmetric Two-Neighbor Constraint

- Let $A^{asym}(n,k) \subset S_n$ consists of all permutations that satisfy the asymmetric two-neighbor k-constraint.
- Let $C^{asym}(\epsilon) = \lim_{n\to\infty} \frac{\log |A^{asym}(n,k)|}{\log n!}$, $k = n^{\epsilon}$.

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Theorem: $C^{asym}(\epsilon) = 1$.

Lower Bound on $|A^{asym}(n,k)|$

- For a set I, let $I^{\nearrow}$ be the increasing ordering of I, and let $I^{\searrow}$ be the decreasing ordering of I.

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Example $I = \{10,2,8,4,3,9\}$.

$$I^{\nearrow} = 2,3,4,8,9,10$$

$$I^{\searrow} = 10,9,8,4,3,2$$

Lower Bound on $|A^{asym}(n,k)|$

- Let $\pi \in C_{r,2}$.

$$\pi = \rho(\gamma_1, \gamma_2, \ldots, \gamma_r), \qquad \gamma_i \in S(2i-1, 2i)$$

$$\rho = [\sigma(1), \sigma(1), \sigma(2), \sigma(2), \ldots, \sigma(r), \sigma(r)], \qquad \sigma \in S_r$$

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Example: $r = 2, \ \ \sigma = [3,1,2],$

$$\gamma_1 = [2,1], \gamma_2 = [3,4], \gamma_3 = [6,5].$$

$$\rho = [3,3,1,1,2,2]$$

$$\pi = [6,5,2,1,3,4].$$

Lower Bound on $|A^{asym}(n,k)|$

- Let $r \leq \frac{n-1}{4}$ and let $C_r \subset S_n$ s.t. $\sigma \in C_r$ if there exist:

1) A partition of $\{2r+1, 2r+2, \dots, n\}$ into $2r+2$ classes $I_1, I_2, \dots, I_{2r+2}$ (nonempty, except I_{2r+2}).
2) A permutation $\pi \in C_{r,2}$.

and $\sigma = [{I_1}^{\nearrow}, {I_2}^{\searrow}, \pi(1), \pi(2), {I_3}^{\nearrow}, {I_4}^{\searrow}, \pi(3), \pi(4), \dots,$
$\pi(2r-1), \pi(r), {I_{2r+1}}^{\nearrow}, {I_{2r+2}}^{\searrow}]$

Lower Bound on $|A^{asym}(n,k)|$

Example: $n = 15, r = 2,$

$$I_1 = \{5,8,10\}, \qquad I_2 = \{6,12\},$$
$$I_3 = \{7\}, \qquad I_4 = \{9,15\},$$
$$I_5 = \{13\}, \qquad I_6 = \{11,14\}$$
$$\pi = [4,3,1,2] \in C_{2,2}$$

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$$\sigma = [\underbrace{5,8,10}_{I_1^{\nearrow}},\underbrace{12,6}_{I_2^{\searrow}},\underbrace{4,3}_{\pi(1),\pi(2)},\underbrace{7}_{I_3^{\nearrow}},\underbrace{15,9}_{I_4^{\searrow}},\underbrace{1,2}_{\pi(3),\pi(4)},\underbrace{13}_{I_5^{\nearrow}},\underbrace{14,11}_{I_6^{\searrow}}]$$

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$S(\ell,k)$- Stirling number of second kind.

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Lower Bound on $|A^{asym}(n,k)|$

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$$C_n{}^{asym} = \bigcup_{r=0}^{\frac{n-1}{4}} C_r,$$

- $C_n{}^{asym} \subset A^{asym}(n,1)$ and

$$|C_n{}^{asym}| \geq \sum_{r=0}^{\frac{n-1}{4}} \frac{1}{2}(2r)!\,S(n-2r,2r)$$

Capacity of Asymmetric Two-Neighbor Constraint Codes

- Let $r = \lceil \delta n \rceil$, $0 < \delta < \frac{1}{4}$.

$$\frac{\log |A^{asym}(n,1)|}{\log n!} \geq \frac{\log |C_n{}^{asym}|}{\log n!} \geq \frac{\log |C_r|}{\log n!}$$

$$\geq \frac{\log \frac{1}{2}(2r)!\, S(n-2r, 2r)}{\log n!} \geq 1 - \delta.$$

Capacity of Asymmetric Two-Neighbor Constraint Codes

- Let $r = \lceil \delta n \rceil$, $0 < \delta < \frac{1}{4}$.

$\log|A^{asym}(n,1)| \geq \log|C_n{}^{asym}|$

$$C^{asym}(\epsilon) \geq C^{asym}(0) = 1$$

Conclusion

- Capacity of two-neighbor constrained codes.
- Capacity of two-neighbor constrained error-correcting codes, with the inversion metric.
- Capacity of asymmetric two-neighbor constrained codes.

Thank You!