Combinatorial Systematic Switch codes

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Reference


Background

Problem Formulation

Consecutive-Generation Switch Codes

Switch Codes Serving Other Requests
Background

- A network switch is a computer networking device that connects devices together in a computer network.
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A subsystem of a network switch, called switch memory, is used to temporarily cache data packets between their arrival from input ports and departure at output ports.
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A subsystem of a network switch, called switch memory, is used to temporarily cache data packets between their arrival from input ports and departure at output ports.

Try to improve its data exchange speed by applying coding on the network switches.
Network Switches

Switch Memory Subsystem

- Input Ports
- Write Controller
- Memory Banks
- Read Controller
- Output Ports
Data Flow

- **Write Controller**
  - Input Ports: 2
  - Memory Banks: 2

- **Read Controller**
  - Output Ports: 2
Data Flow with Conflict

Write Controller

Input Ports: 2

Memory Banks: 2

Output Ports: 2

Read Controller

A

C

D

B

C

A

D

B

Input Ports: 2

Memory Banks: 2

Output Ports: 2
Solution by Replication

Write Controller

Output Ports: 2

Memory Banks: 4

Input Ports: 2

Read Controller

A

C
Solution by Coding²

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Switch Codes Serving Other Requests
Problem Formulation

- $R_{in}$: number of input ports
- $R_{out}$: number of output ports
- $n$: number of memory banks
- Redundancy: $n - R_{in}$
- Generation: data packets written in the same time slot
Example

\[ R_{in} = R_{out} = 2, \ n = 3, \ \text{Redundancy} = 1, \ \text{two generations.} \]
Problem Formulation: Encoding

**Encoding**: A function $f$ from generations of input data packets to written bits in memory banks

$$f : \Sigma^{R_{in}} \longrightarrow \Sigma^{n}$$

$$U \longmapsto X,$$

where $U = (u_{i,1}, u_{i,2}, \ldots, u_{i,R_{in}})$ and $X = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$ are input and written bits of the $i$-th generation.
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Let $(u_{i_1,l_1}, u_{i_2,l_2}, \ldots, u_{i_{R_{out}},l_{R_{out}}})$ be arbitrary $R_{out}$ request bits from previous generations.
Request Bits: \((u_{i_1}, l_1, u_{i_2}, l_2, \ldots, u_{i_{R_{\text{out}}}}, l_{R_{\text{out}}})\)

Request vector: \(L = (l_1, l_2, \ldots, l_{R_{\text{out}}})\)

Partition of \(L, L_i, 1 \leq i \leq t:\) denotes request sets from the \(i\)-th generation.

Example

The request bits are \((u_{1,1}, u_{1,2}, u_{2,3}, u_{3,3})\), then \(L = (1, 2, 3, 3)\), \(L_1 = \{1, 2\}\), \(L_2 = \{3\}\), \(L_3 = \{3\}\).

Solution: A way to read request bits by retrieving no more than one bit from each memory bank.
Different type of requests

Request vector: \( L = (l_1, l_2, \ldots, l_{R_{\text{out}}}) \)

- \( t \) consecutive-generation request
- In the worst case, the request bits are all from different generations.

1. **limited-repetition request**: each integer repeats at most a certain number of times in \( L \).
2. **one burst request**: only one integer appear more than once in one request vector (**Locally recoverable codes**).
3. the request vector consists of all multiset (**Batch Codes**).
Background

Problem Formulation

Consecutive-Generation Switch Codes

Switch Codes Serving Other Requests
Definitions

- **Request Bits:** \((u_1, l_1, u_2, l_2, \ldots, u_{R_{out}}, l_{R_{out}})\)
- **Request vector:** \(L = (l_1, l_2, \ldots, l_{R_{out}})\)
- **Partition of** \(L\), \(L_i, 1 \leq i \leq t\): denotes request sets from the \(i\)-th generation.

A **\(t\)-consecutive-generation switch code:** \((n, R_{in}, R_{out}; t)\) switch code

*The \(t\)-consecutive-generation switch codes are applicable when the data packets expire after a fixed time slot and only the newest \(t\) generations are of interest.*
For \( t = 2 \):

**Theorem (Wang et al. (2013))**

There is an \((n, R_{in}, R_{out}; 2)\) switch code with redundancy \(R_{in} - 1\) if \(R_{out} \leq 2R_{in} - 1\).

**Proof.**

Take a path on \( R_{in} \) vertices, encoding with this path.

\[
(u_{i,1}, u_{i,2}, \ldots, u_{i,R_{in}}) \Rightarrow
(u_{i,1}, u_{i,2}, \ldots, u_{i,R_{in}}, u_{i,1} + u_{i,2}, u_{i,2} + u_{i,3}, \ldots, u_{i,R_{in} - 1} + u_{i,R_{in}})
\]
Example

\( R_{in} = 4: \)

\[
\begin{bmatrix}
  u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} & u_{2,1} + u_{2,2} & u_{2,2} + u_{2,3} & u_{2,3} + u_{2,4} \\
  u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & u_{1,1} + u_{1,2} & u_{1,2} + u_{1,3} & u_{1,3} + u_{1,4}
\end{bmatrix}
\]
For general $t$:

**Lemma**
A complete graph on $n$ vertices contains $\lfloor n/2 \rfloor$ edge-disjoint Hamiltonian paths.

**Theorem**
There is an $(n, R_{in}, R_{out}; t)$ switch code with redundancy $(R_{in} - 1)(t - 1)$, provided

$$t - 1 \leq \lfloor R_{in}/2 \rfloor \text{ and } t - 1 \leq R_{in} - \lfloor R_{out}/t \rfloor.$$
Proof

\[ \sum R_{in} \longrightarrow \sum R_{in} + (R_{in} - 1)(t-1) \]

Let \( G \) be a subgraph of the complete graph on \( R_{in} \) vertices, formed by the edges of \( t - 1 \) edge-disjoint hamiltonian paths, where \( t - 1 \leq \lfloor R_{in}/2 \rfloor \).

Let the \( t - 1 \) edge-disjoint hamiltonian paths in \( G \) be \( P_1, P_2, \ldots, P_{t-1} \), and let the request sets of the \( t \) consecutive generations be \( L_0, L_1, \ldots, L_{t-1} \).
Proof. Cont

- W.l.o.g, assume that $L_0$ has the smallest size among the $L_i$’s, so that $|L_0| \leq \lfloor R_{out}/t \rfloor \leq R_{in} - (t - 1)$. Therefore we can choose $t - 1$ distinct elements $h_1, h_2, \ldots, h_{t-1}$ from $[1, R_{in}] \setminus L_0$.

- To serve the requests in $L_0$, we directly read the corresponding systematic elements in written bits. To read the requested bits in $L_i$, for $1 \leq i \leq t - 1$, we use systematic bit $h_i$ and edges in the hamiltonian path $P_i$. 
Example

\( R_{in} = 4 \):

\[
\begin{bmatrix}
  u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} & u_{3,1} + u_{3,2} & u_{3,2} + u_{3,3} & u_{3,3} + u_{3,4} \\
  u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} & u_{2,1} + u_{2,2} & u_{2,2} + u_{2,3} & u_{2,3} + u_{2,4} \\
  u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & u_{1,1} + u_{1,2} & u_{1,2} + u_{1,3} & u_{1,3} + u_{1,4} \\
\end{bmatrix}
\]
A lower bound for the redundancy of a systematic pair-parity code is \( n - R_{in} \geq \left\lceil \frac{R_{in}(t-1)}{2} \right\rceil \), when a request is restricted to \( t \) consecutive generations.

When \( 2 \leq t \leq \lfloor R_{out}/2 \rfloor \), the lower bound cannot be achieved.
Background

Problem Formulation

Consecutive-Generation Switch Codes

**Switch Codes Serving Other Requests**
Switch Codes Serving One-Burst Requests

- An \((n, R_{in}, R_{out})\) switch code: not require any information on the generations of request bits.

- A **one-burst request**: only one integer repeats in the request vector.

- Burst request switch codes can be used to clean up the longest request queue among the memory banks and improve the worst case delay.
One-Burst Requests

Example

\[ R_{\text{in}} = R_{\text{out}} = 5, \ L = (1, 1, 1, 3, 4) \]
Example

\{0, 1, 2\}  \{0, 2, 3\}  \{0, 1, 4\}  \{1, 2, 5\}  \{0, 3, 5\}
\{2, 3, 4\}  \{0, 4, 5\}  \{1, 3, 4\}  \{1, 3, 5\}  \{2, 4, 5\}

(16, 6, 6) switch codes for one burst request

\(L = (0, 0, 0, 0, 0, 0)\)
(\(v, \{3\}, 2\))^*-designs

Let \(X\) be \(\{0, 1, \ldots, R_{in} - 1\}\) and \(\mathcal{B}\) be some triples in \(X\) called \textit{blocks};

(C1) Each pair of \(X\) appears in exactly two blocks;

(C2) \(|\mathcal{B}_a \cap \mathcal{B}_b| = 1\) for all distinct \(a, b \in X\), where \(\mathcal{B}_a = \{\{x, y\} : \{a, x, y\} \in \mathcal{B}\}\);

(C3) There do not exist three blocks of the form \(\{a, b, c\}, \{a, b, d\}, \{b, c, d\} \) in \(\mathcal{B}\).
Connections

Theorem

A $(v, \{3\}, 2)^*$-designs gives a solution for any one-burst request for $R_{in} = R_{out} = v$ bits.

- Using sums of blocks in $B$ as parity check bits, we have an $(R + R(R - 1)/3, R, R)$ switch code for any one-burst request.

- Connection to locally repairable codes

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A. Vardy and E. Yaakobi, “Constructions of batch codes with optimal redundancy.”
Connections

Theorem

There exists a \((v, \{3\}, 2)^*\)-design for all \(v \equiv 0, 1 \pmod{6}\), \(v \geq 1116\).

\(\rightarrow (R + R(R - 1)/3, R, R)\) switch code for any one-burst request
Switch Codes Serving Any Requests

Theorem (Wang et al., 2015\textsuperscript{4})

- Simplex code is an \((n, R_{in}, R_{out})\) switch code with \(n = 2^k - 1\), \(R_{in} = k\) and \(R_{out} = 2^{k-1}\).
- Concatenation of the Simplex codes is also a switch code.

\[
\begin{bmatrix}
G & 0 & \cdots & 0 \\
0 & G & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & G
\end{bmatrix}
\]

where \(G\) is the generator matrix of the Simplex code.

Thank you.