Codes and Card Tricks:
Magic for Adversarial Crowds

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Technion Israel Institute of Technology
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Joint work with Sihuang Hu and Ofer Shayevitz

Magic tutorial

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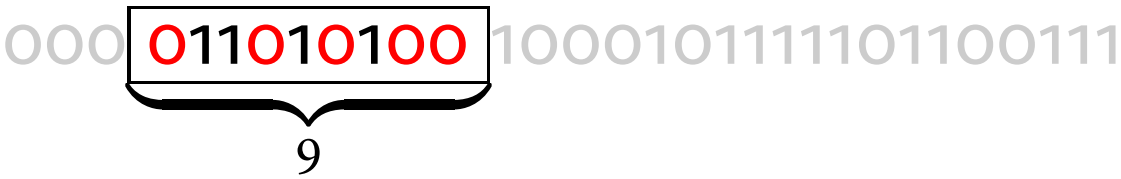

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  - distinct length-9 chunks

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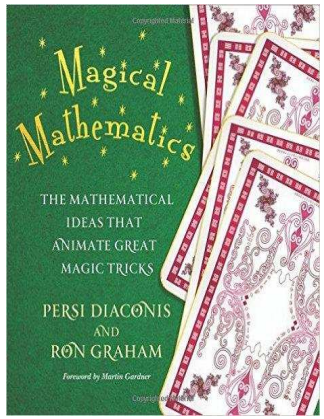
  - Diaconis' mind reader (Diaconis–Graham 2011)
    - ★ Order-5 de Bruijn sequence
    - ★ Top-10 mathematical card trick of the century (Ron Graham)



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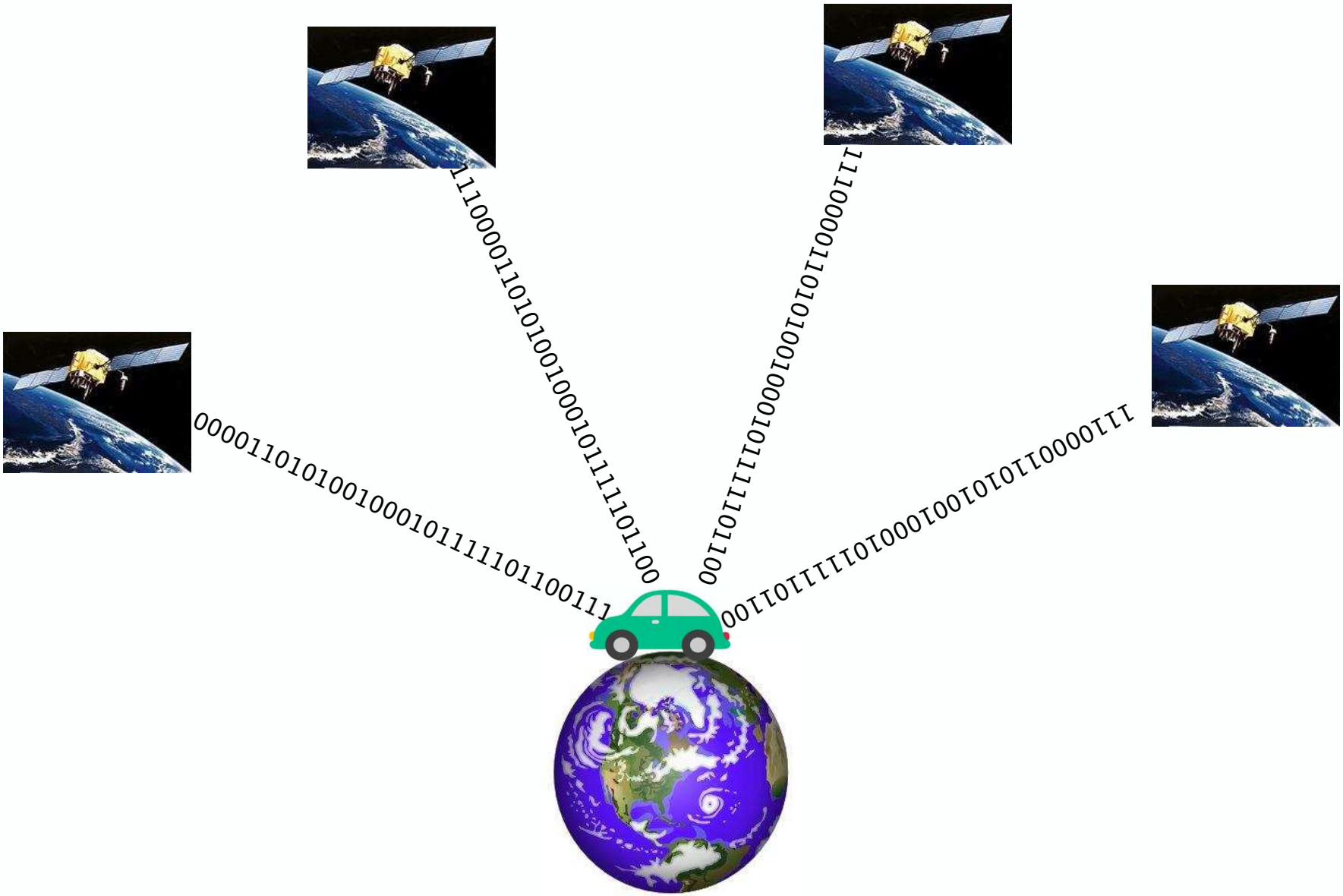

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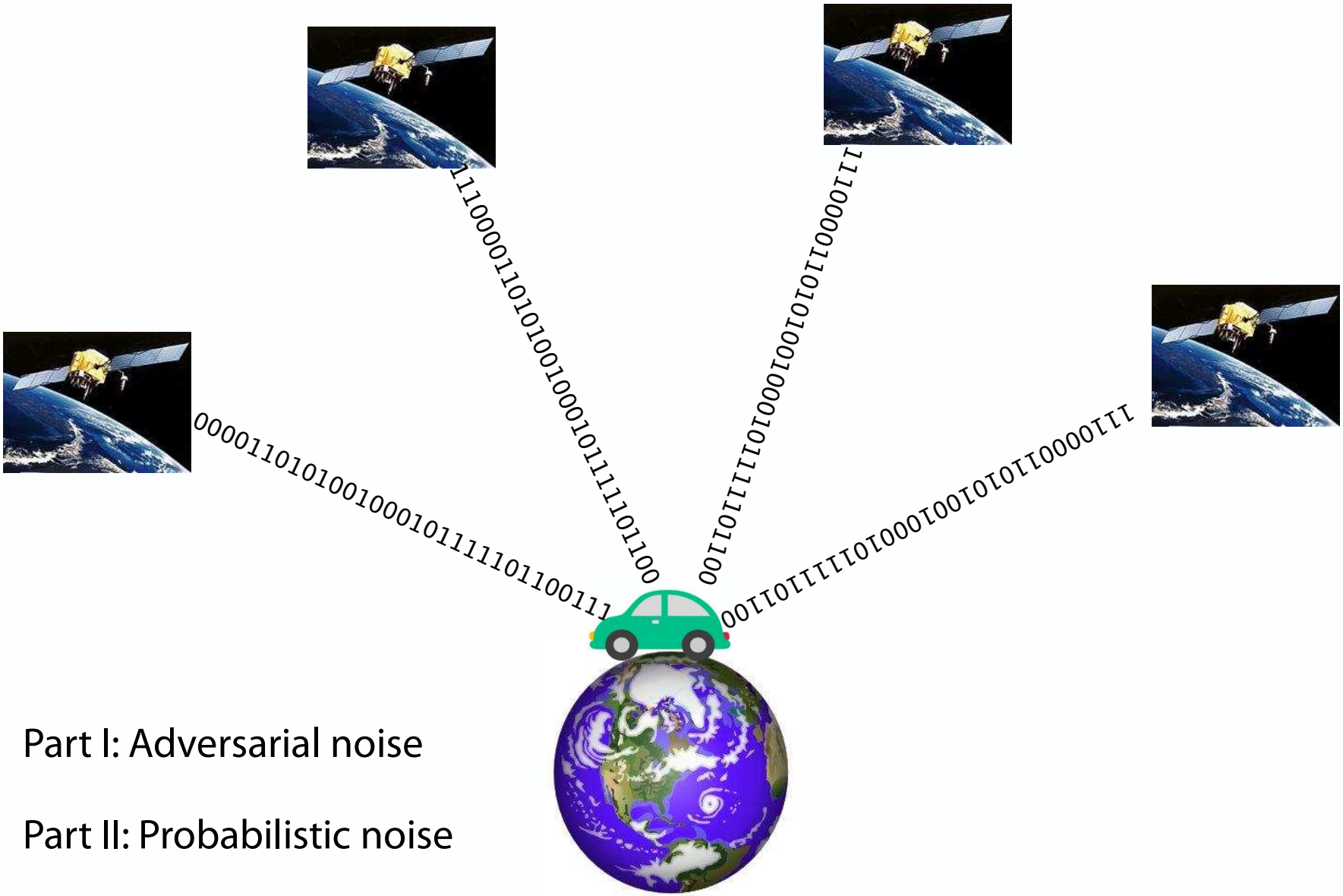

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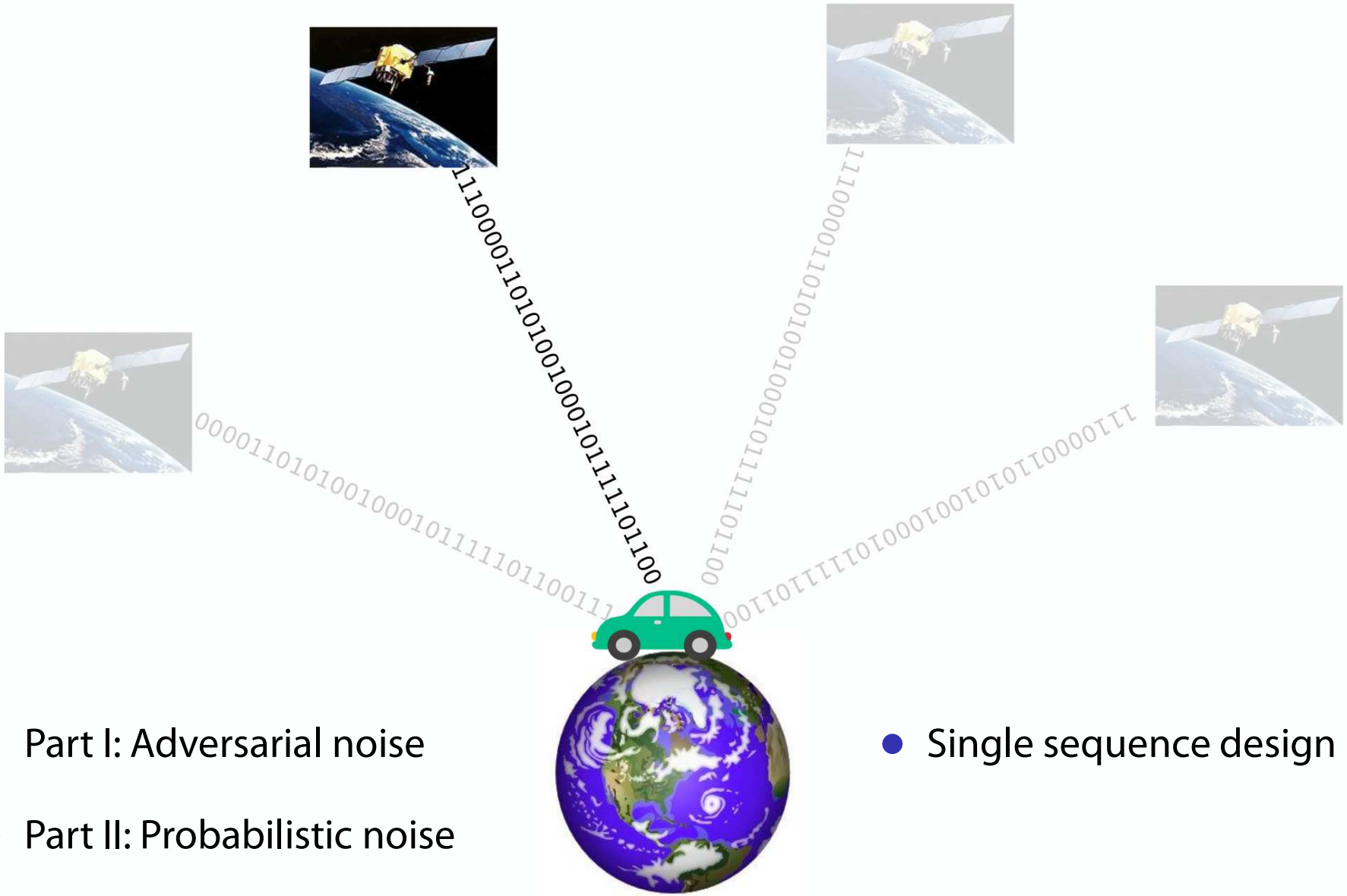

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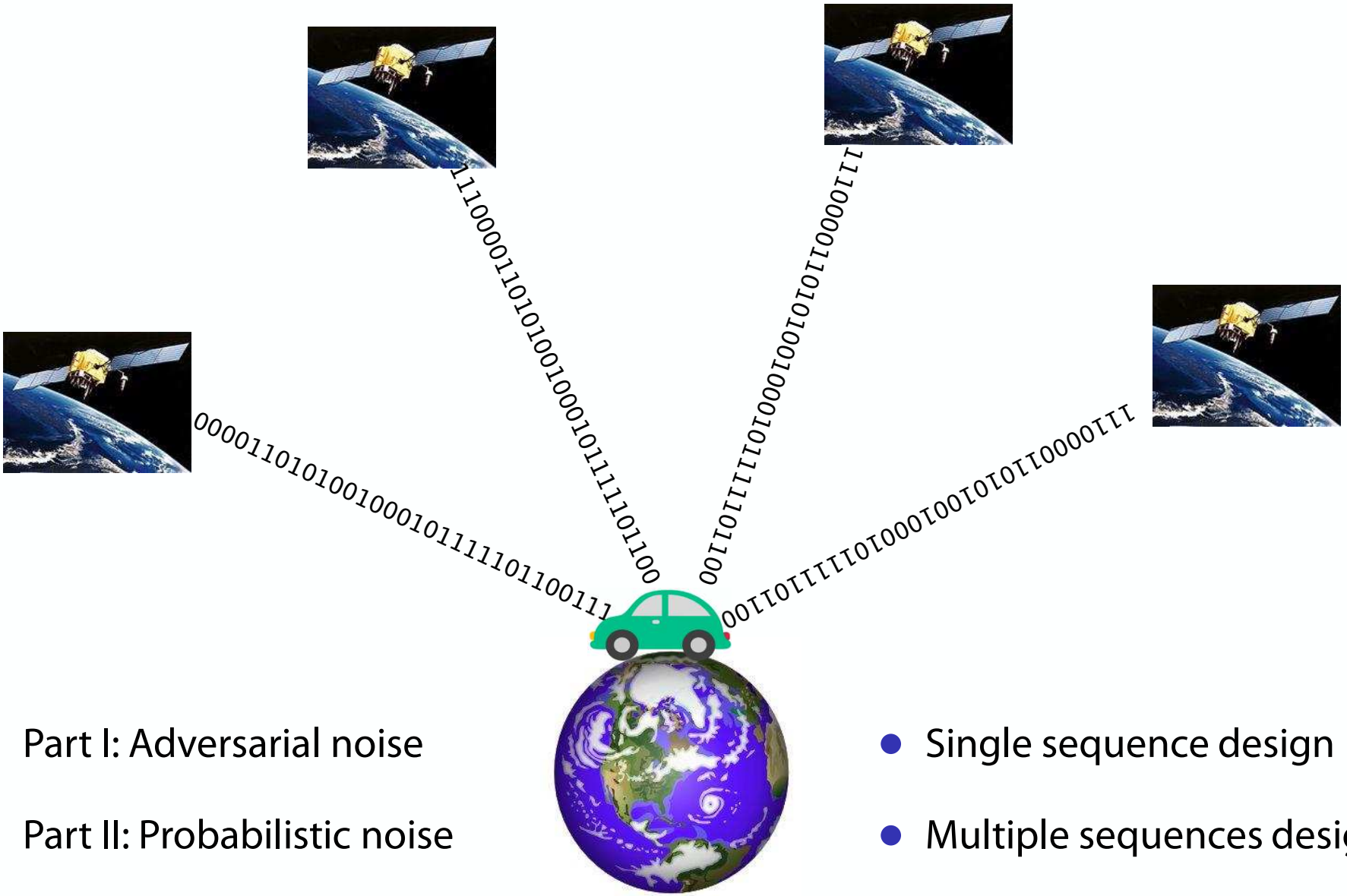

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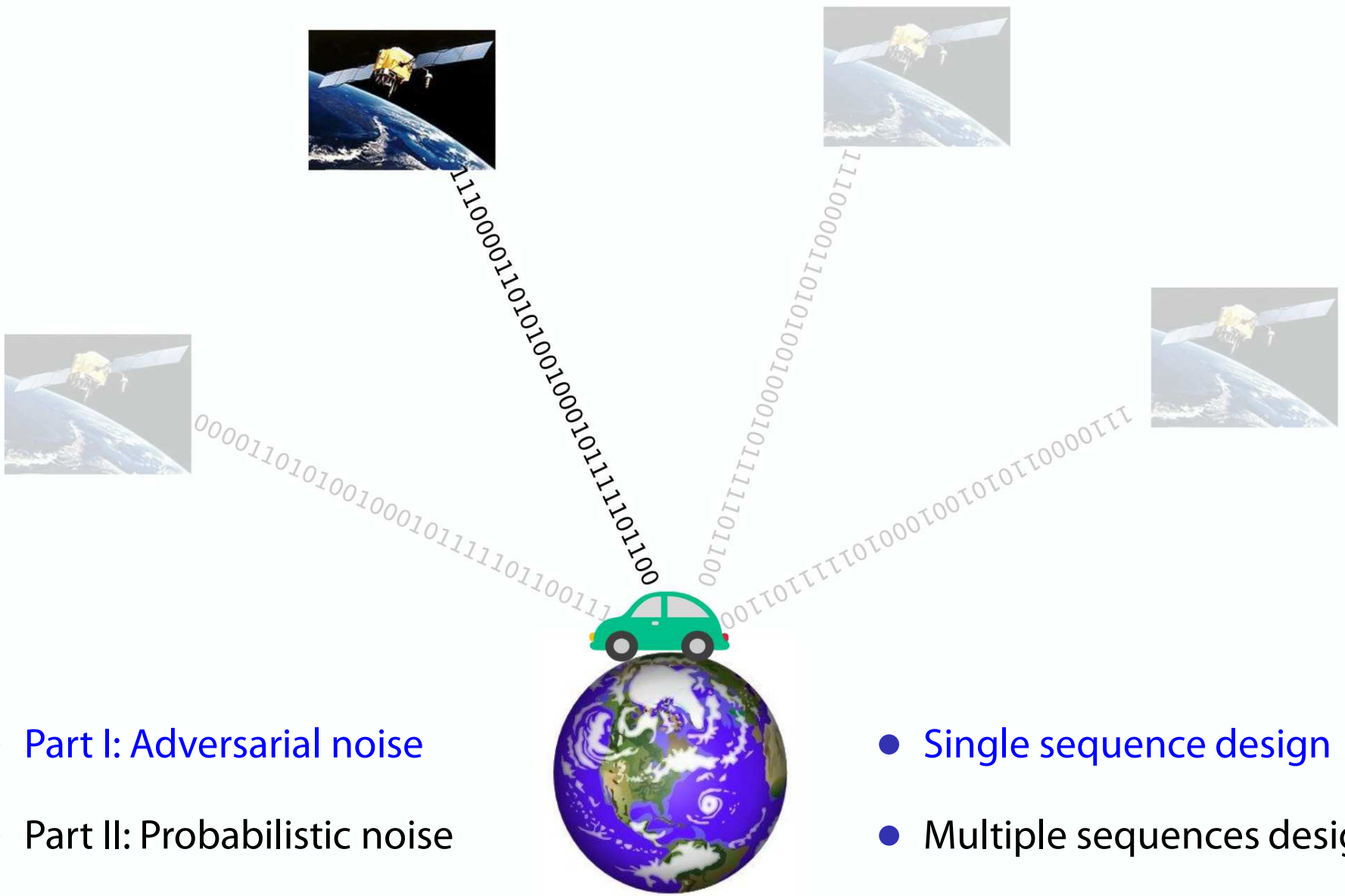

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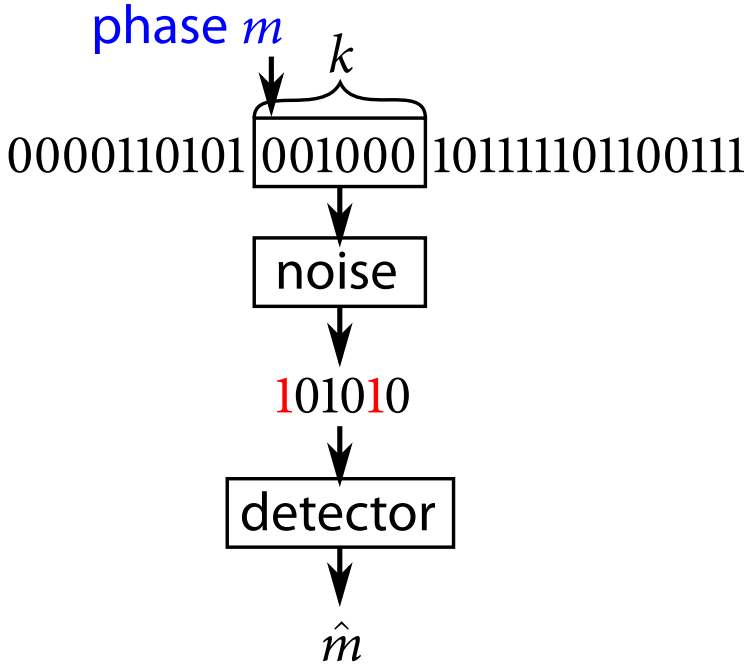

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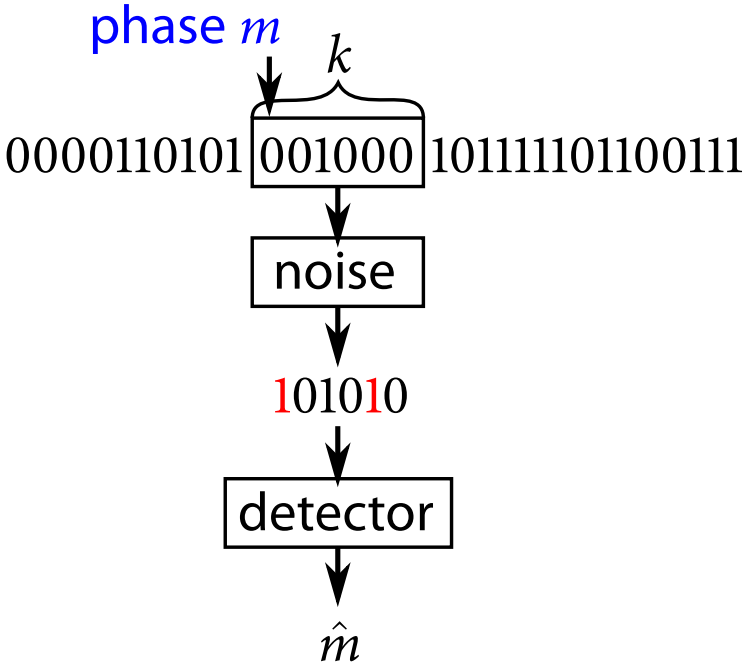

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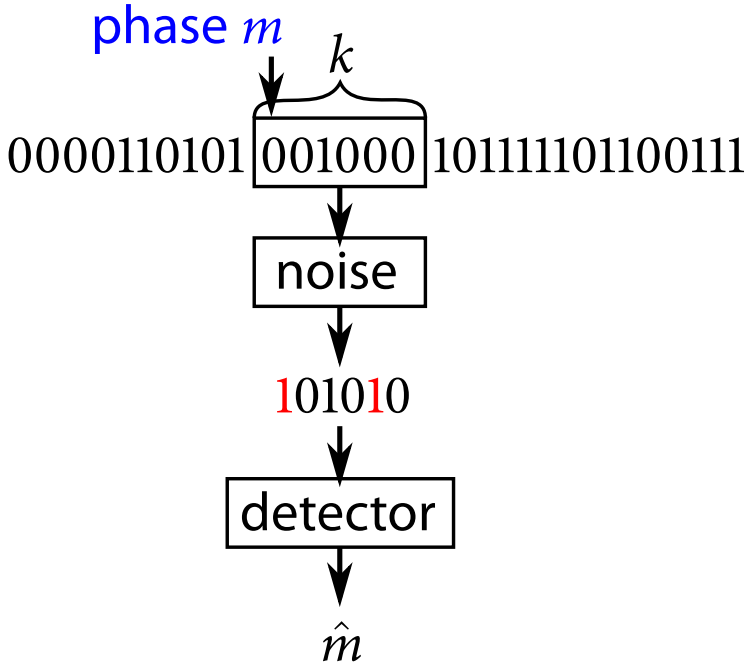

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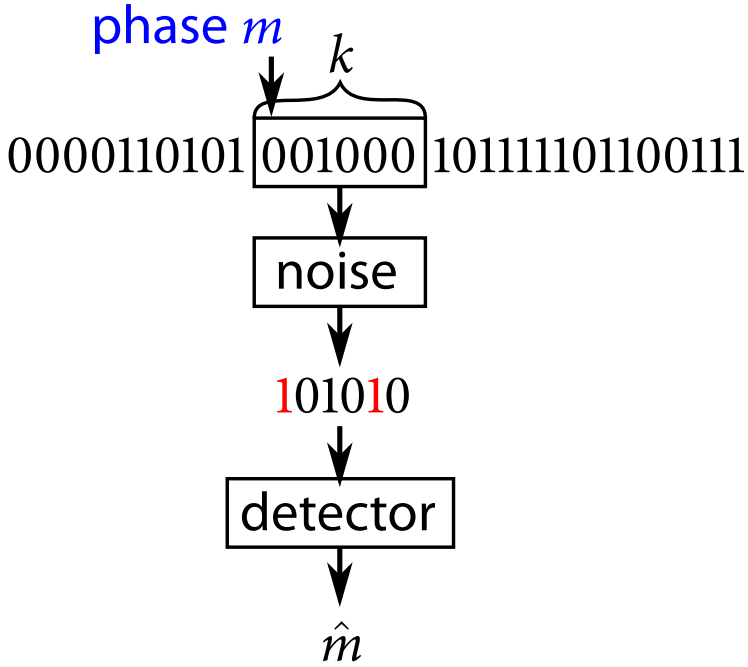

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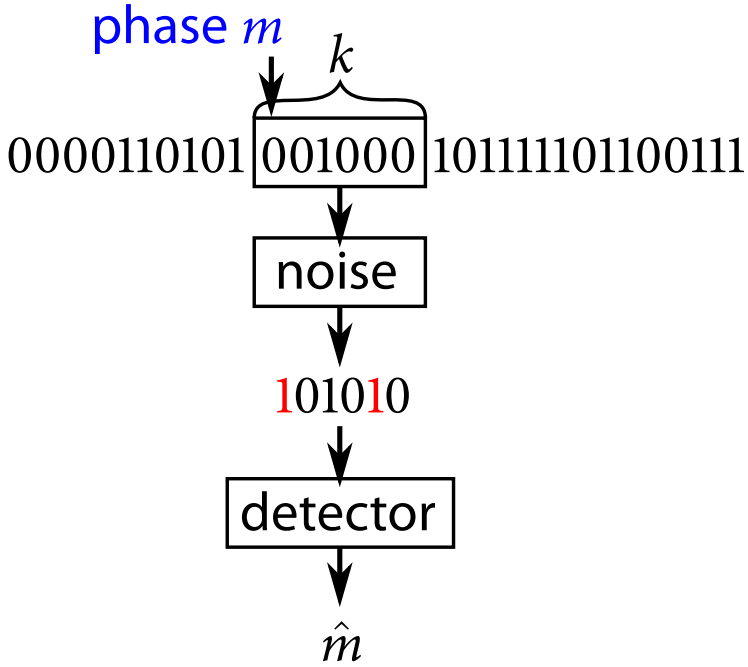

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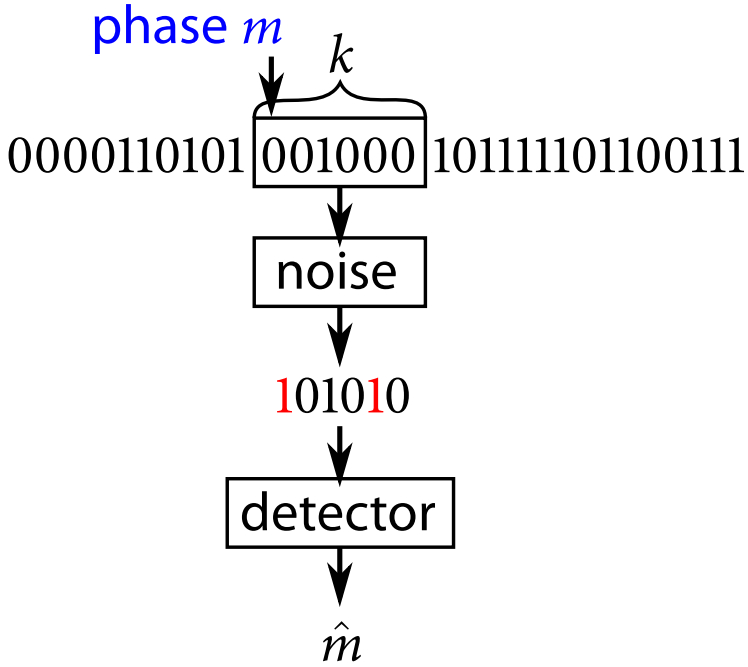

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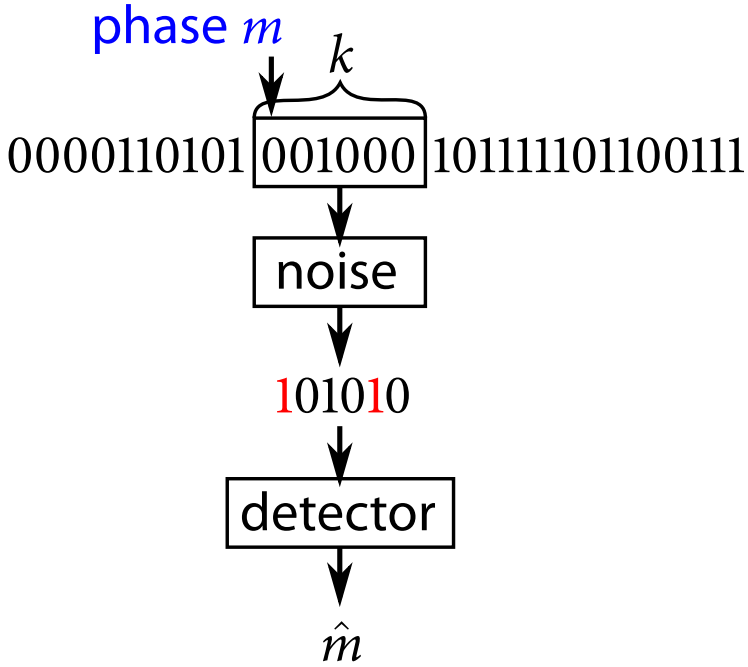

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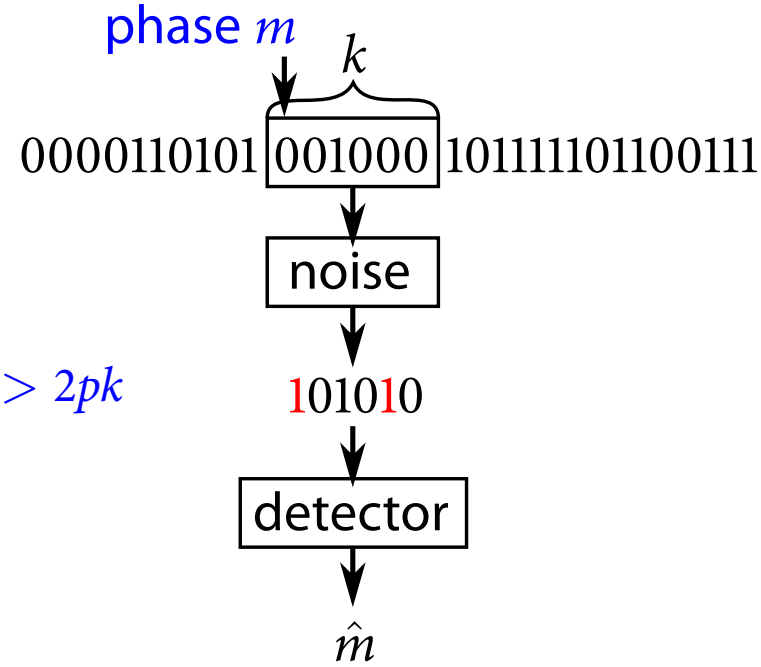

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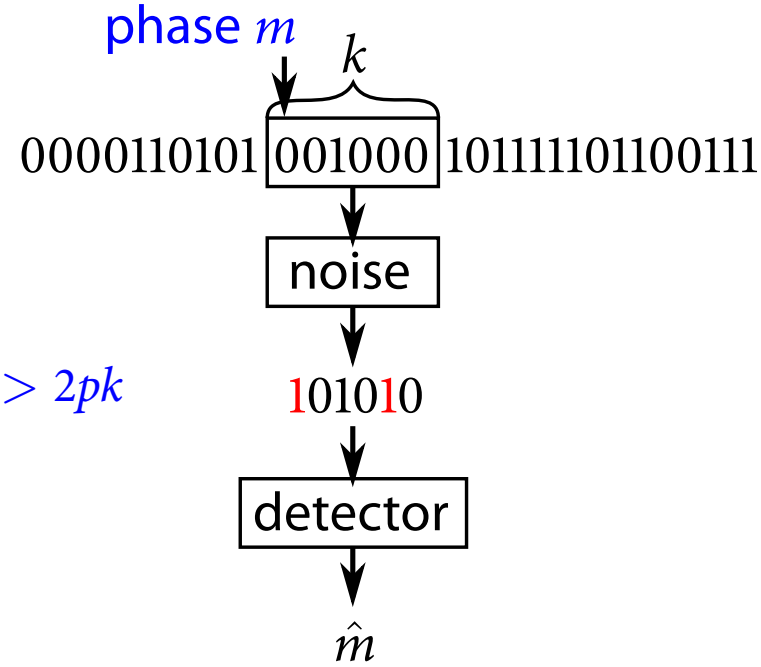

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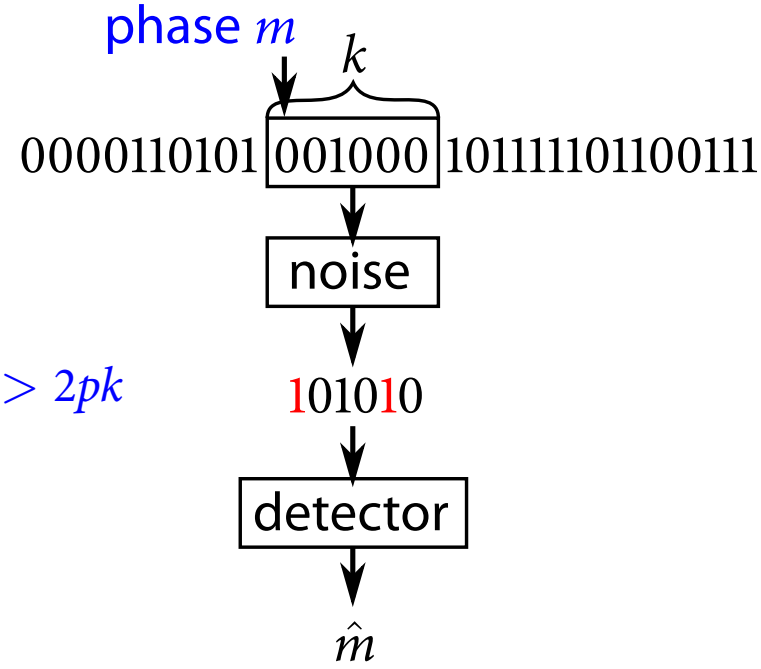

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Relation to channel coding

Phase detection scheme

Error-correcting channel code

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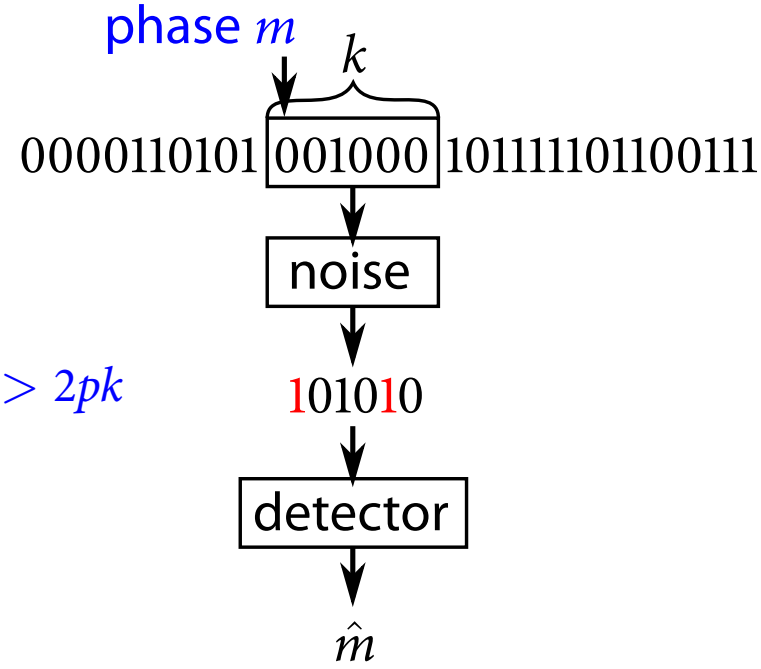

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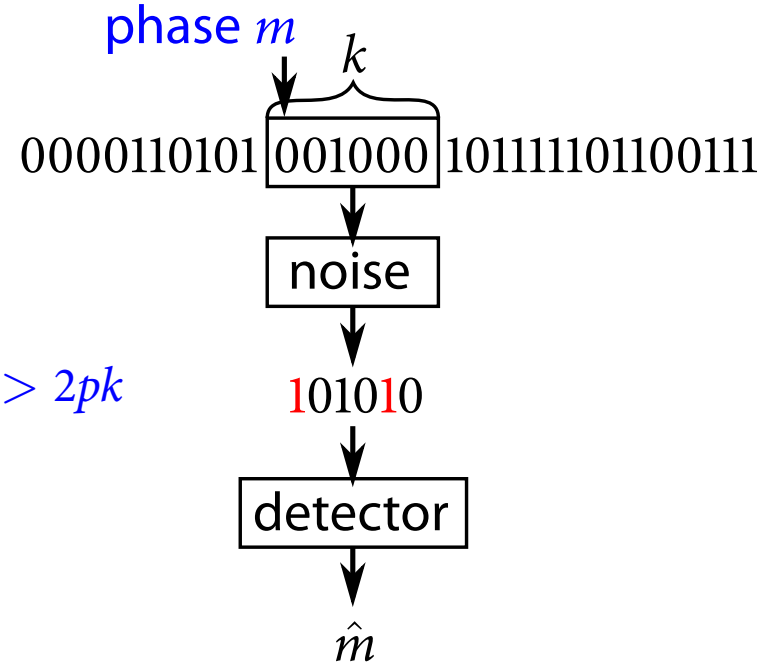

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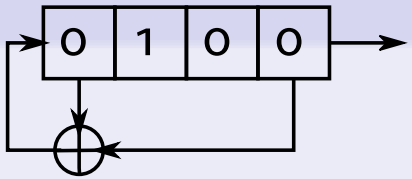

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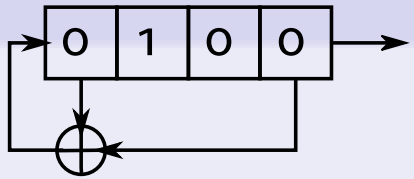

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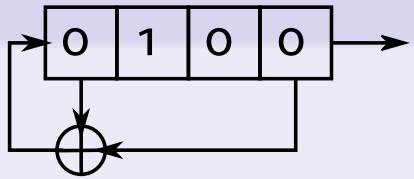

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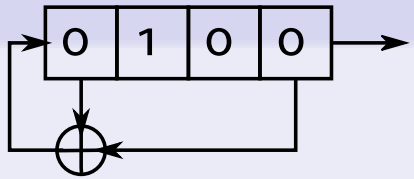

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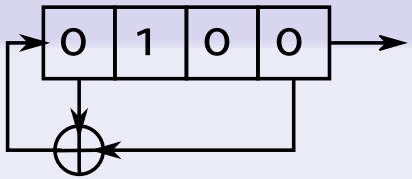

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A linear code can be chained up if and only if

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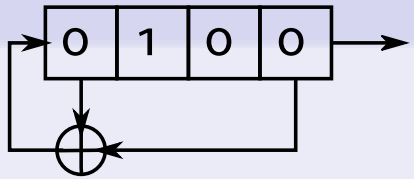

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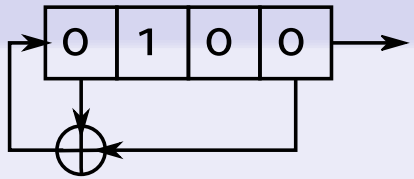

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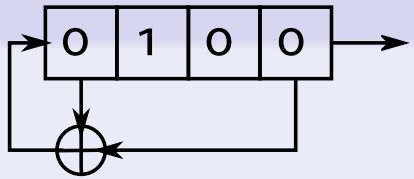

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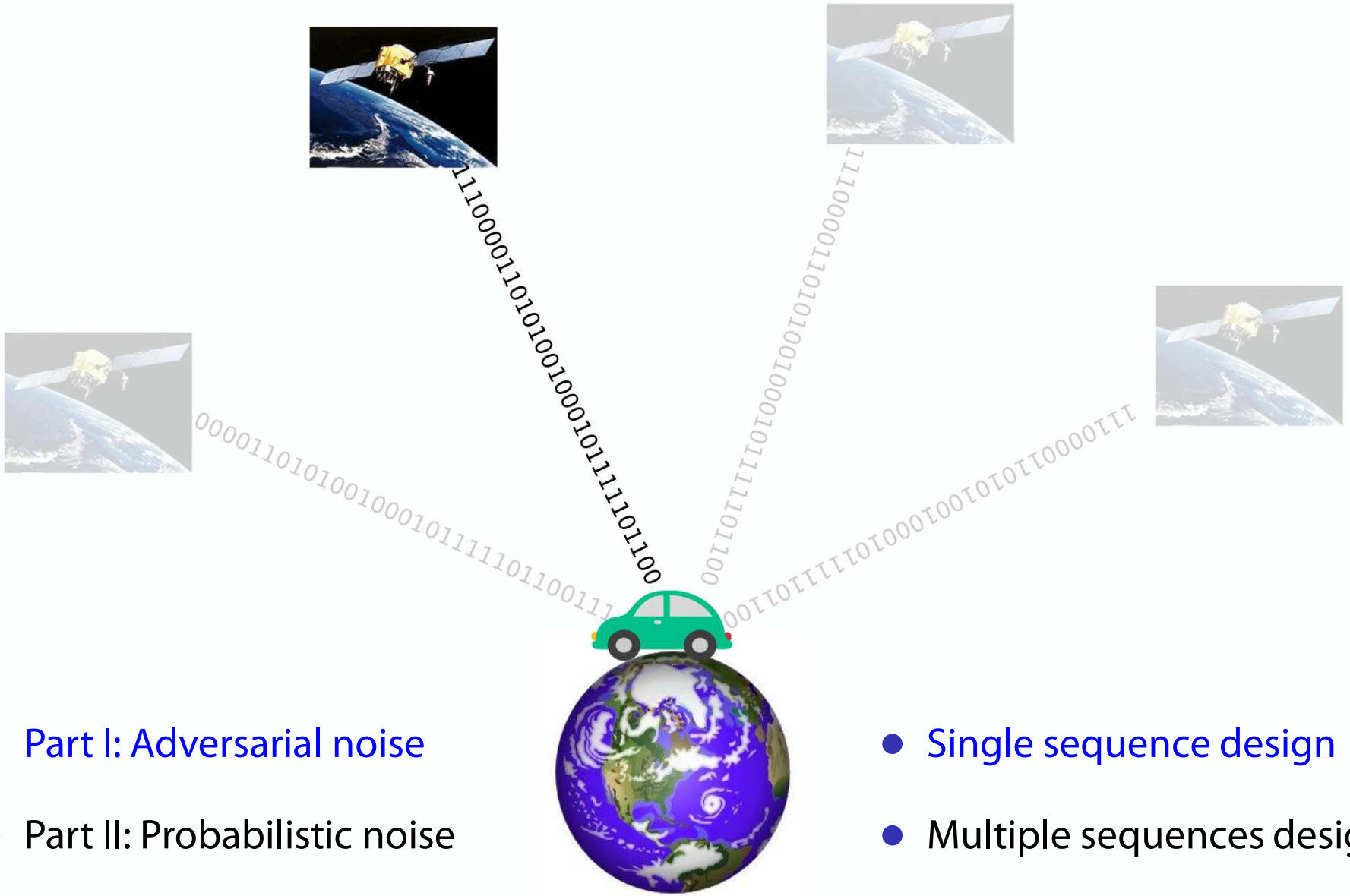

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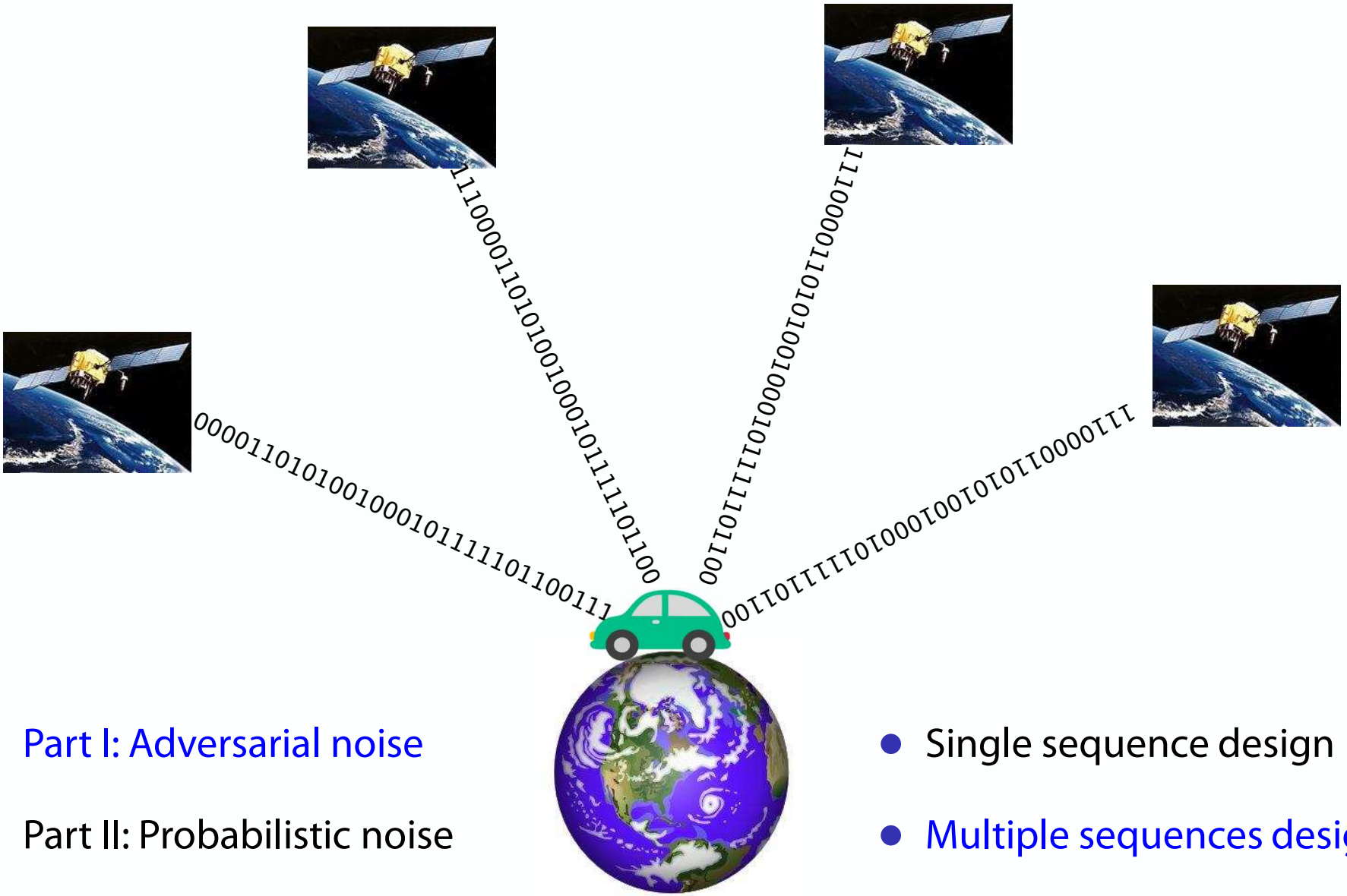

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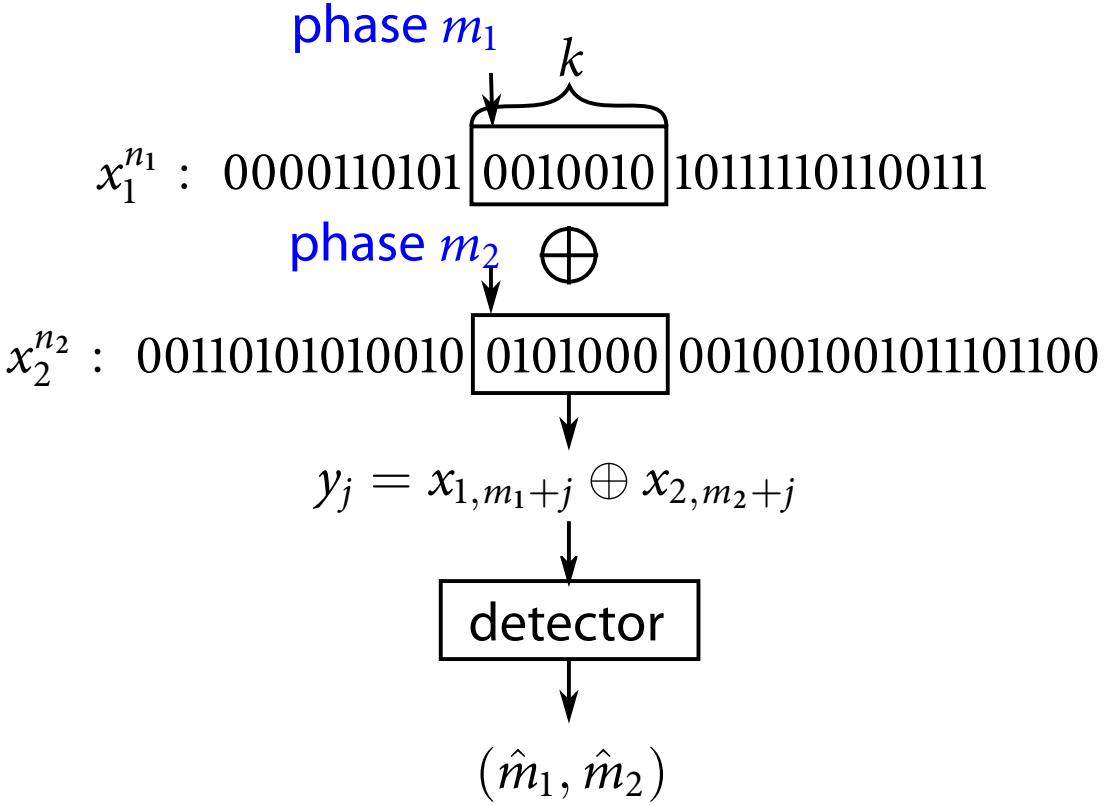

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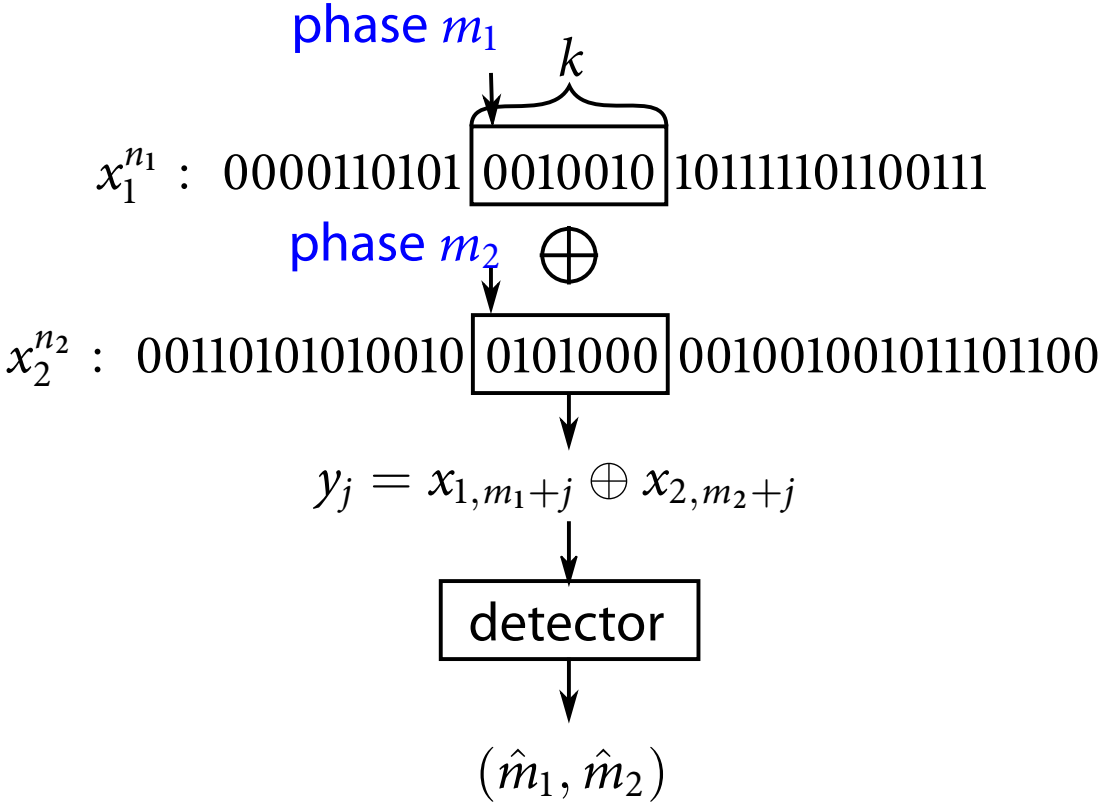

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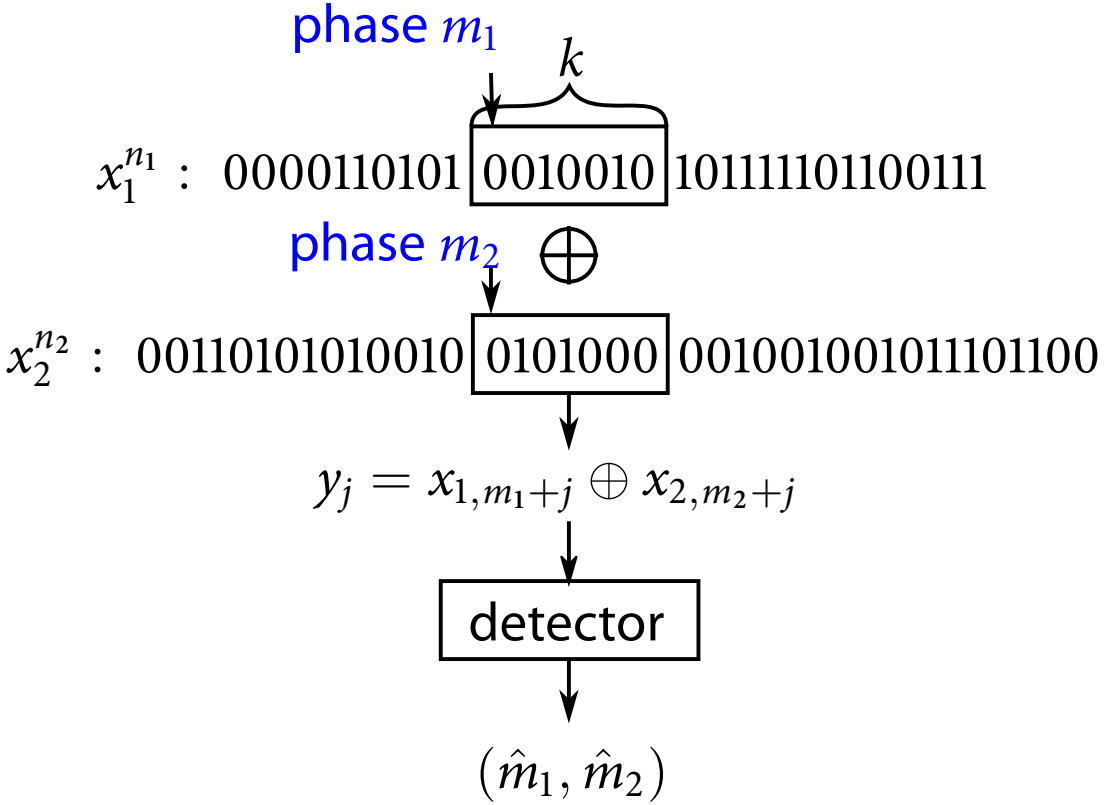

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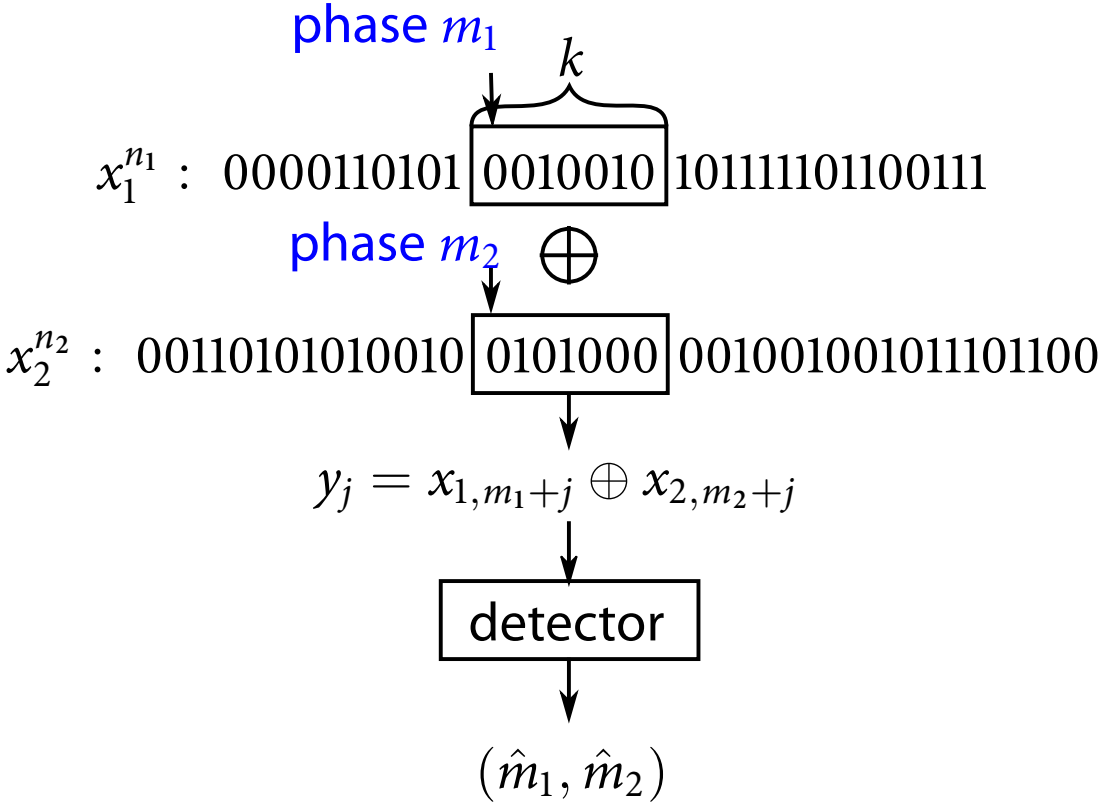

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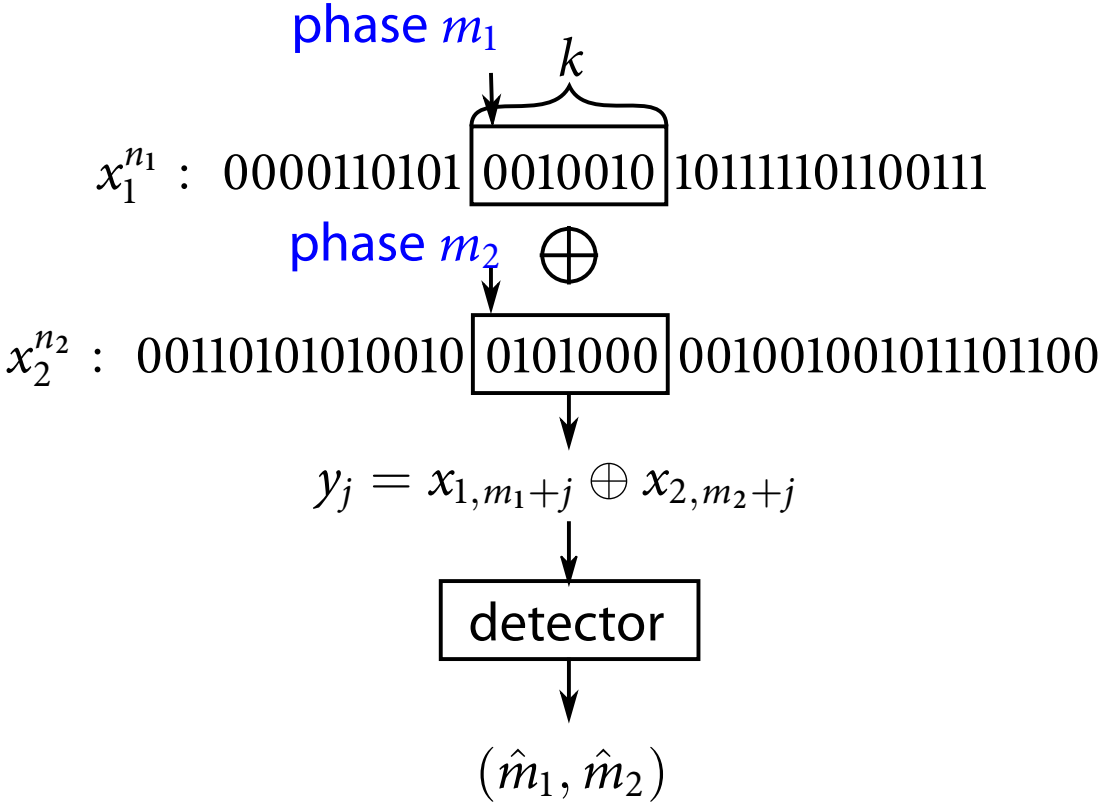

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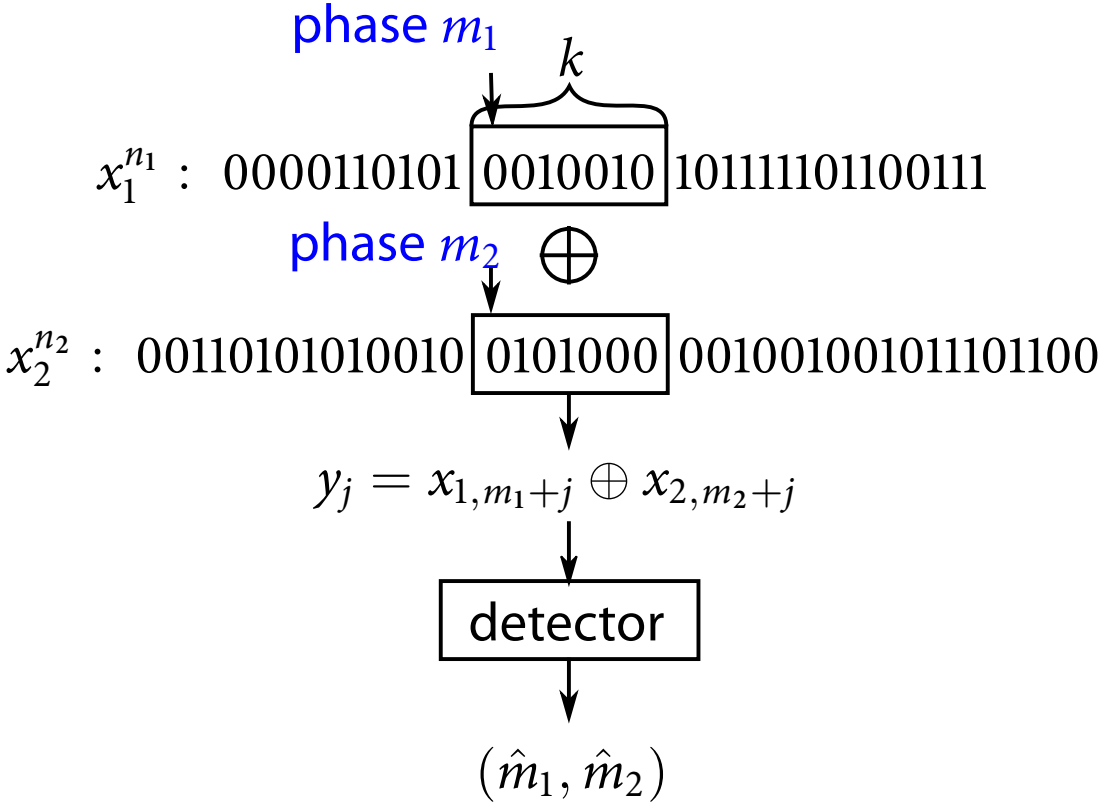

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  - Seq 1 & 2: m-sequences of two distinct primitive polynomials
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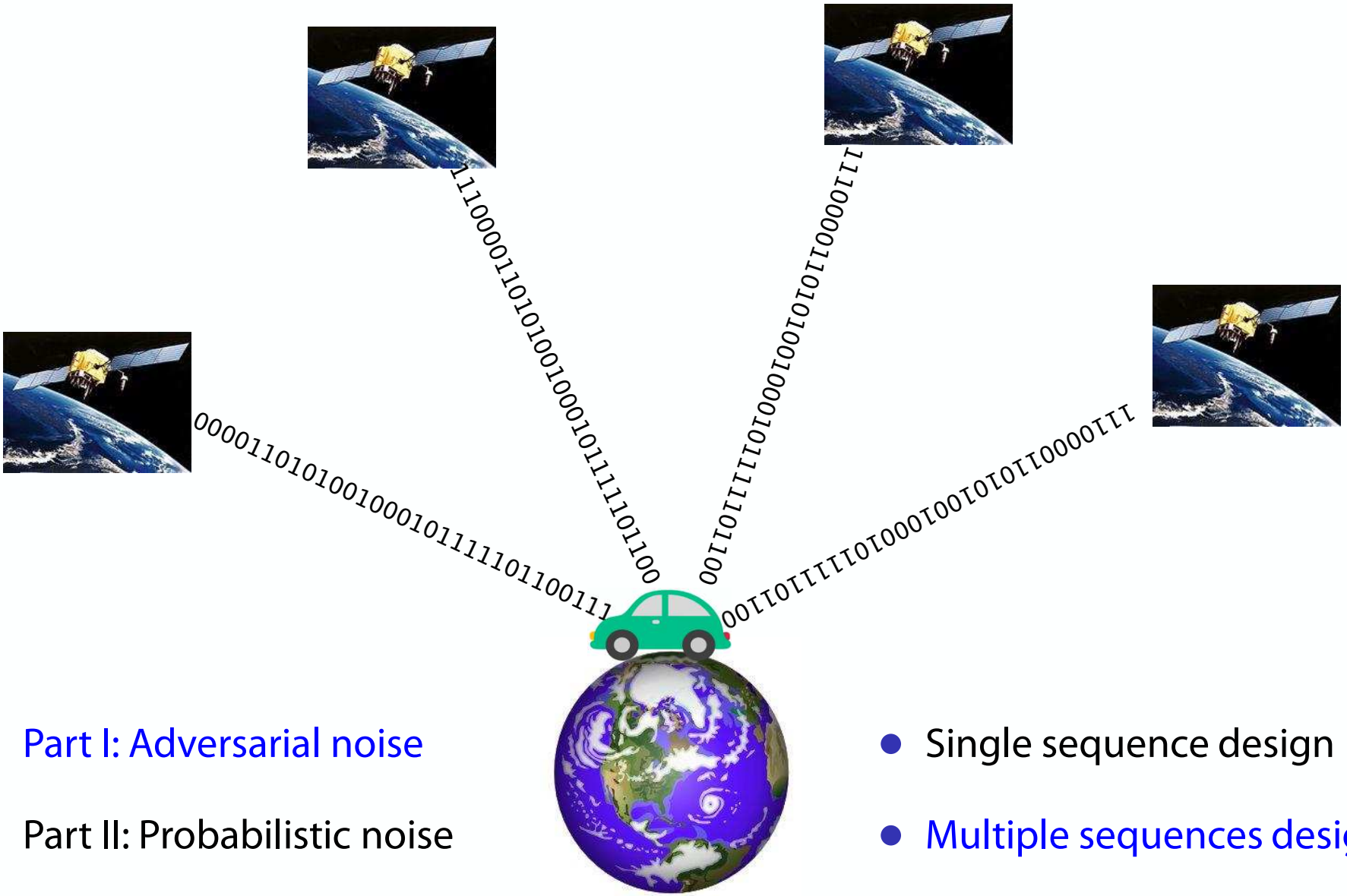

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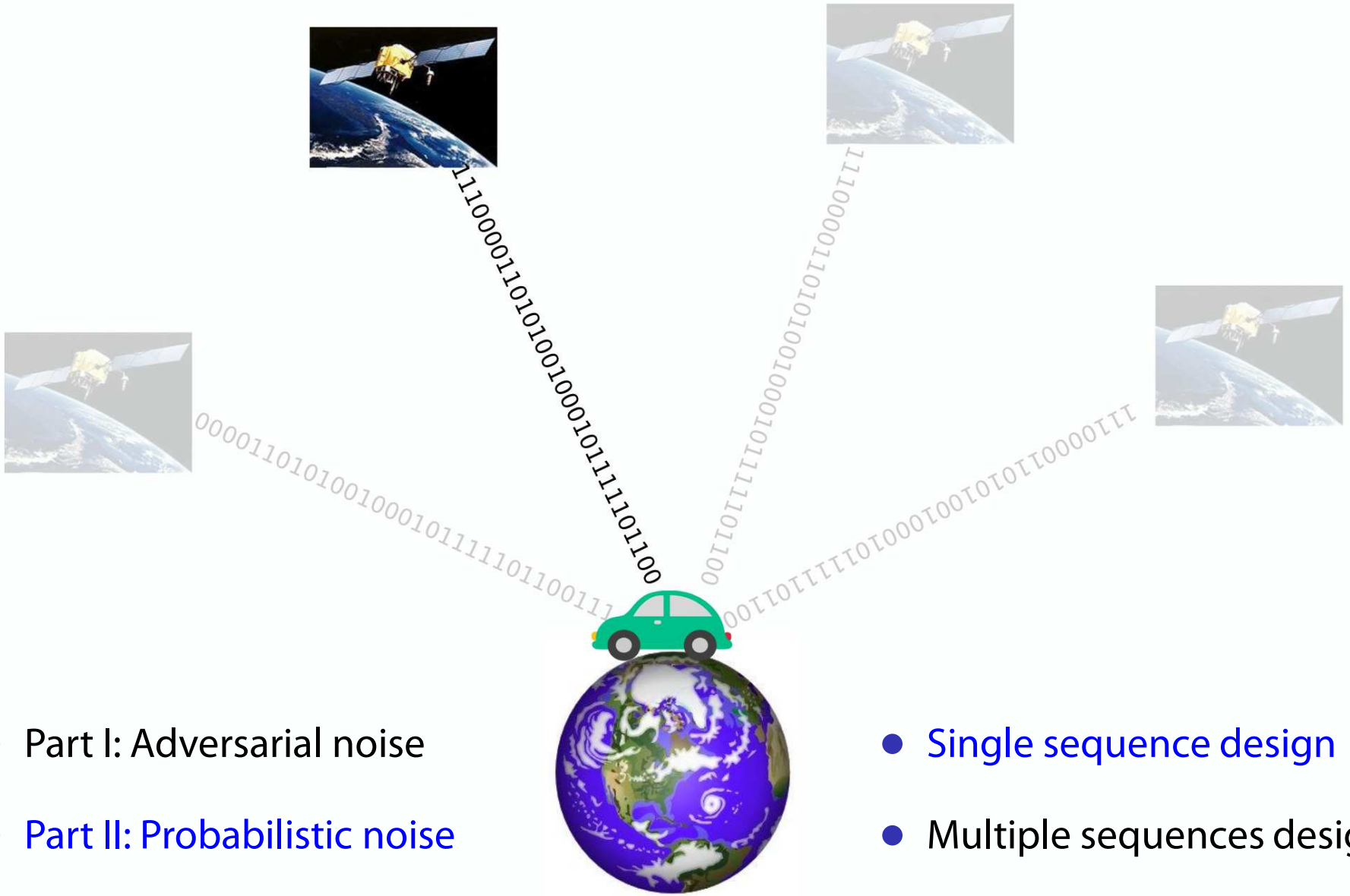

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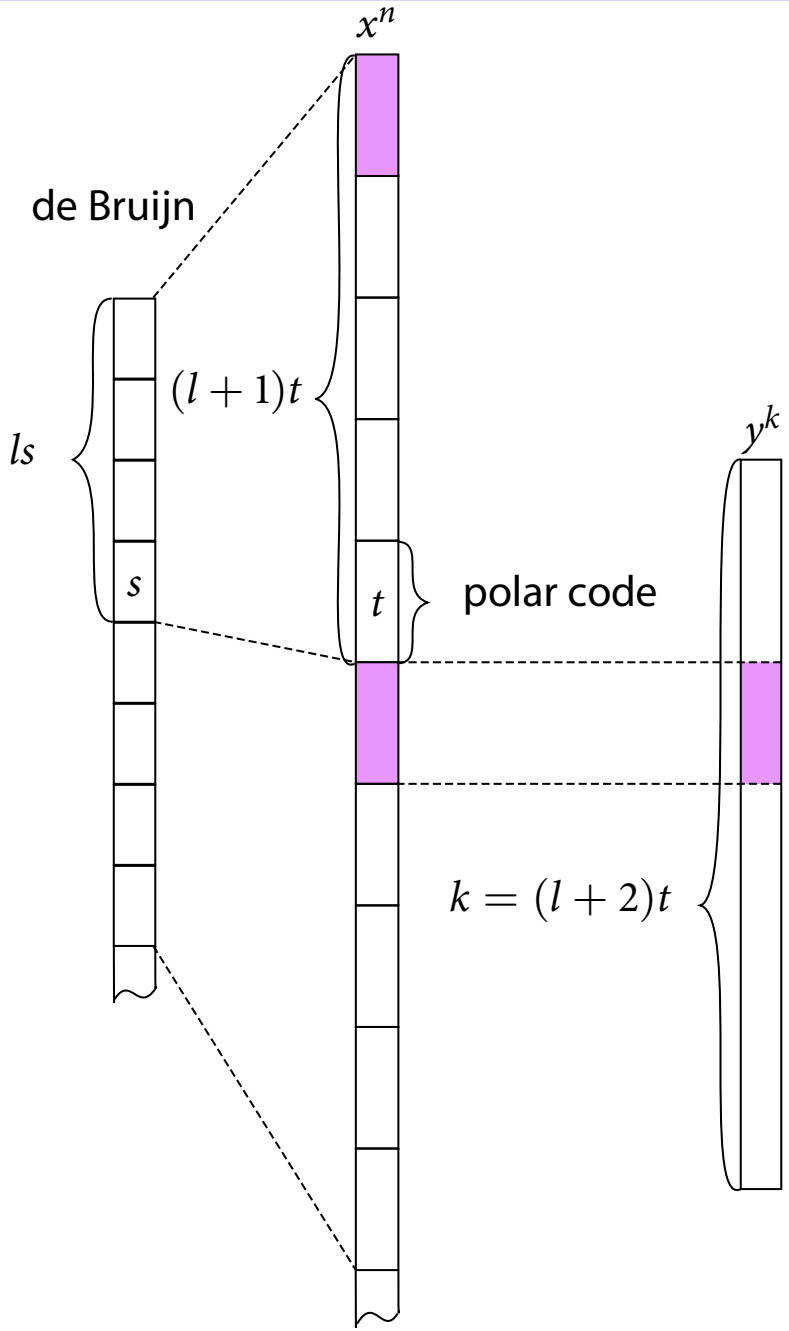
A low-complexity construction

- Capacity achieving
- $O(k \log k)$ complexity



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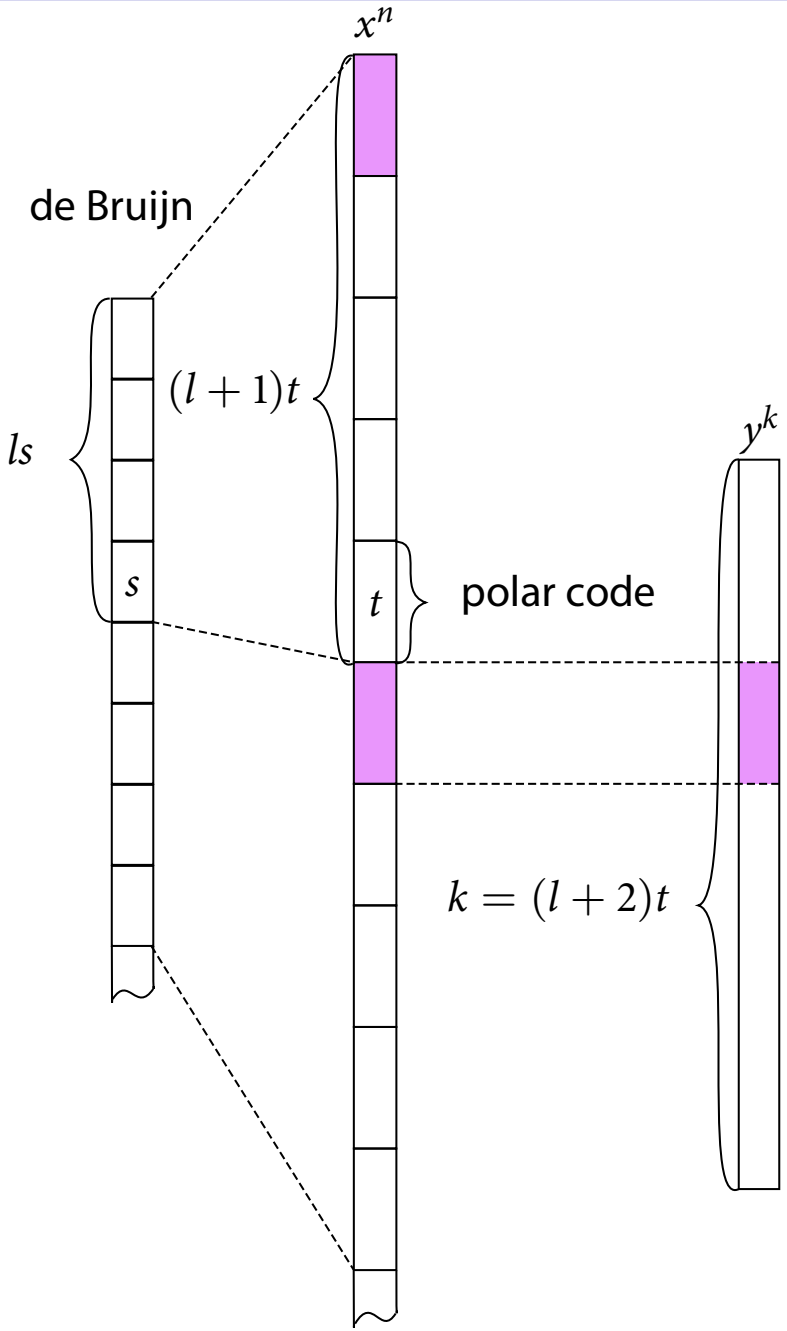

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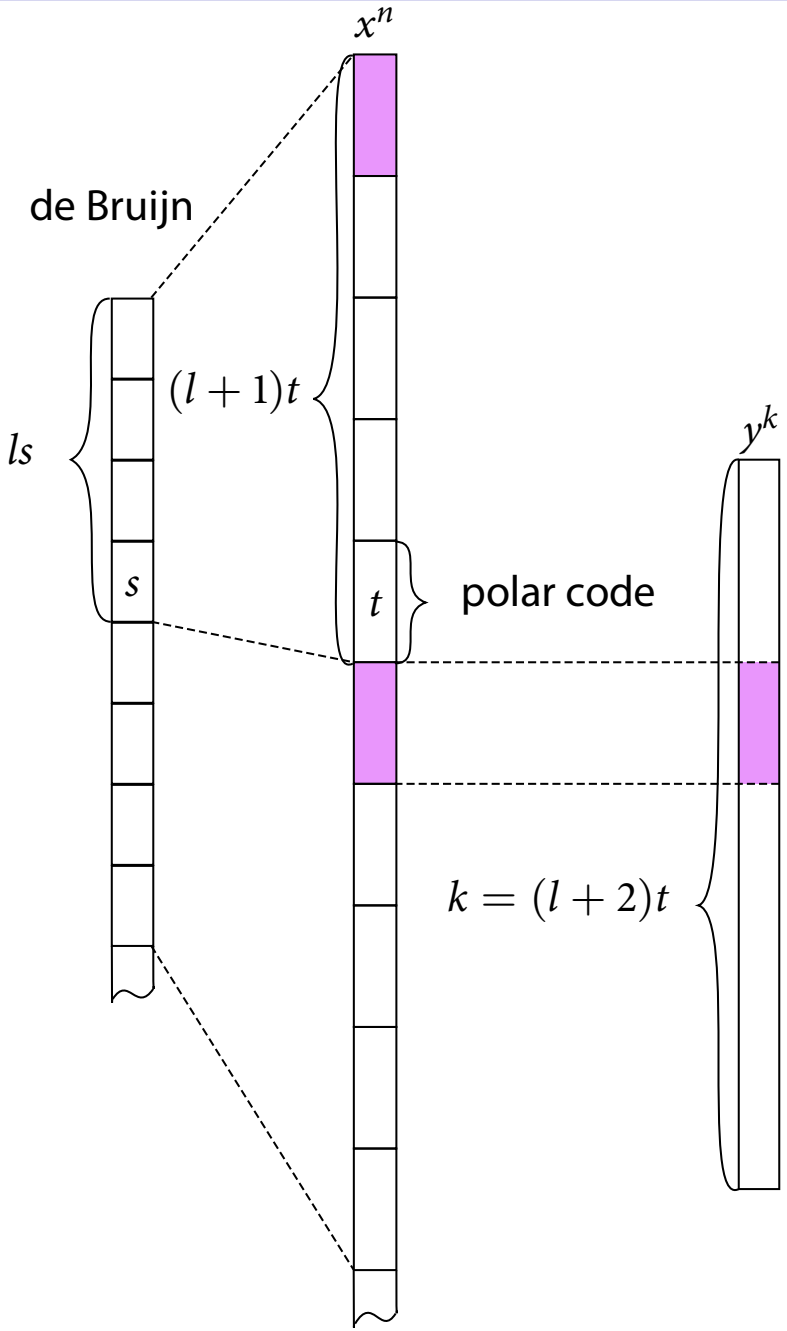

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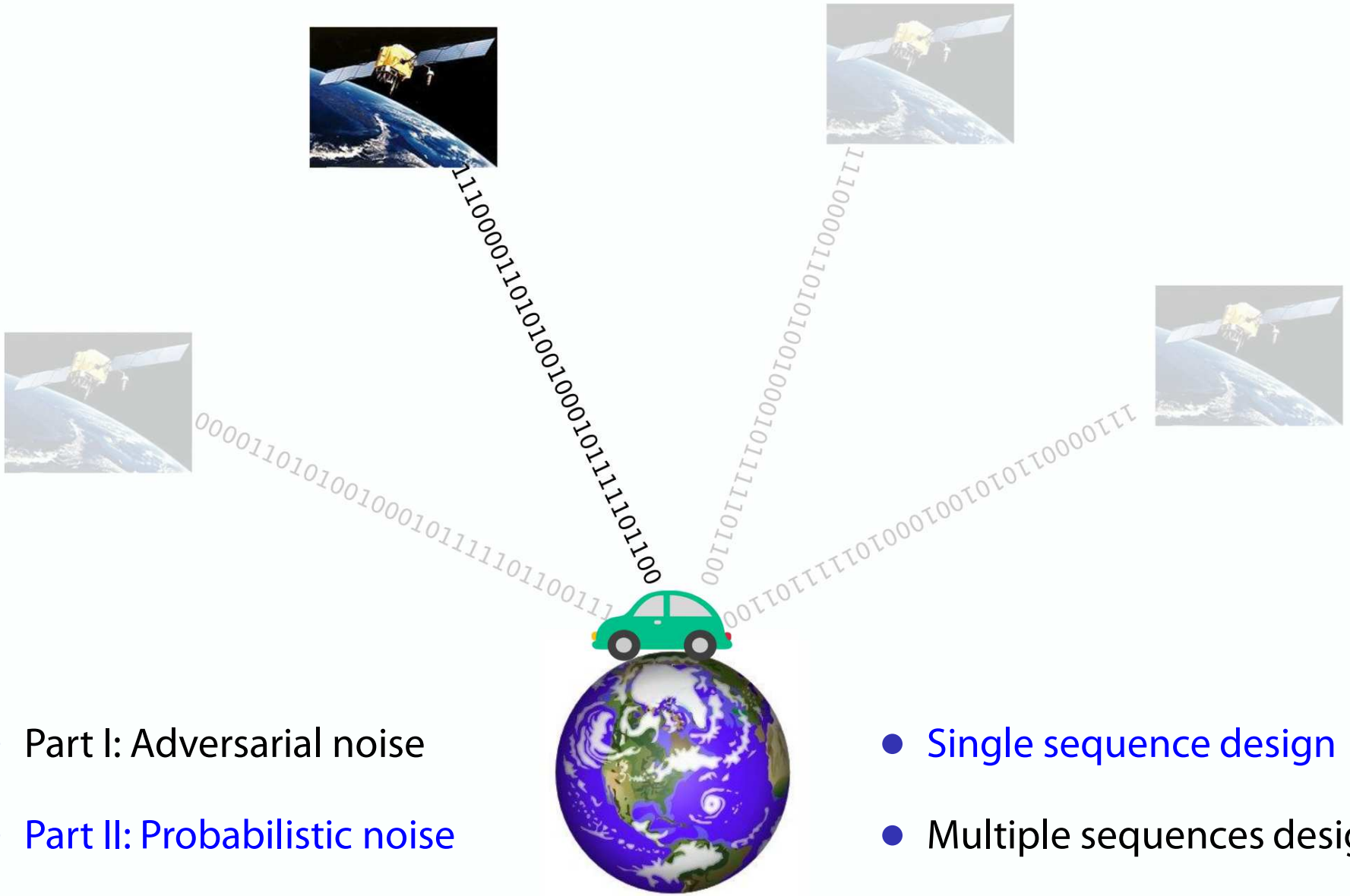

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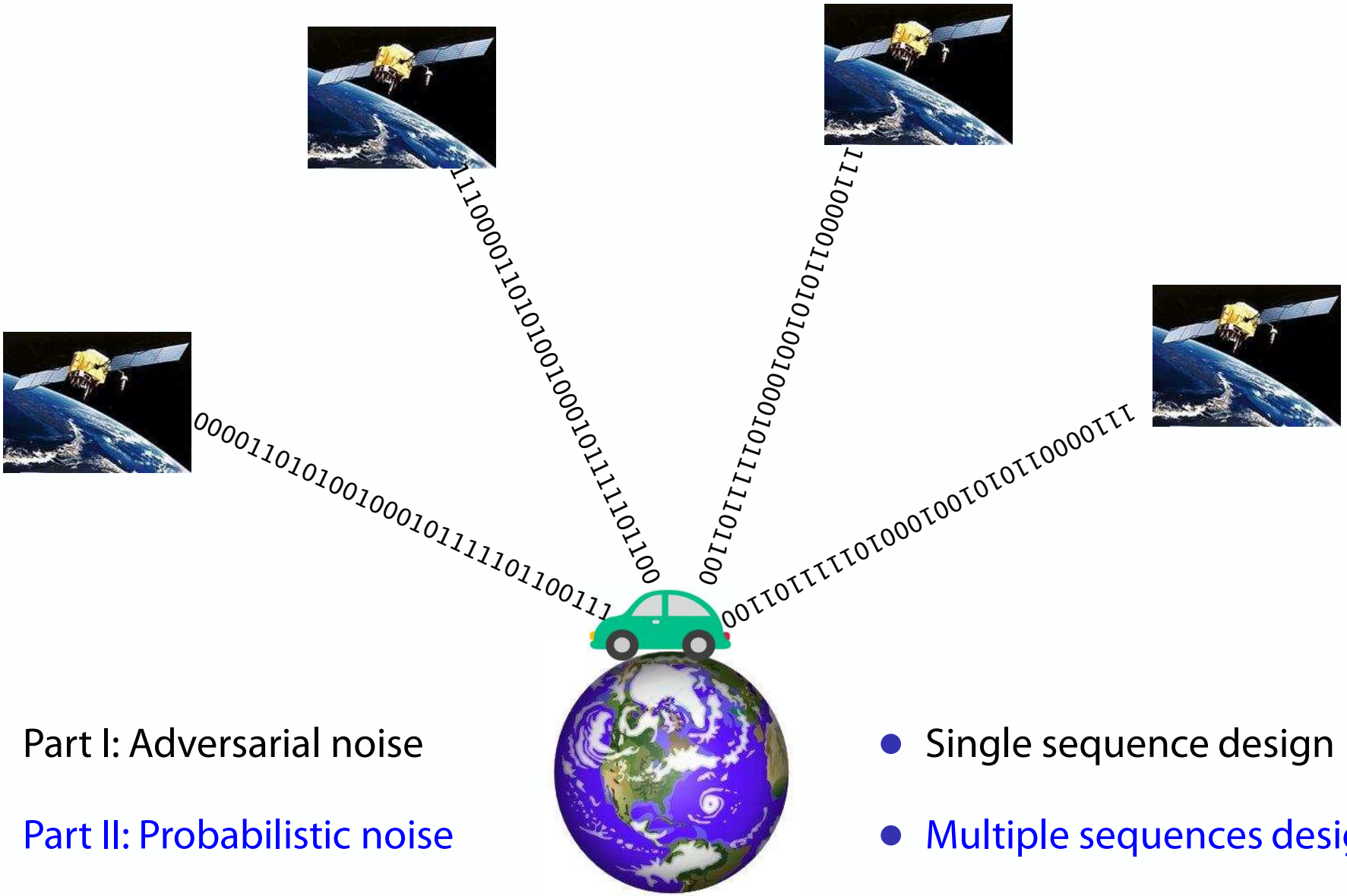

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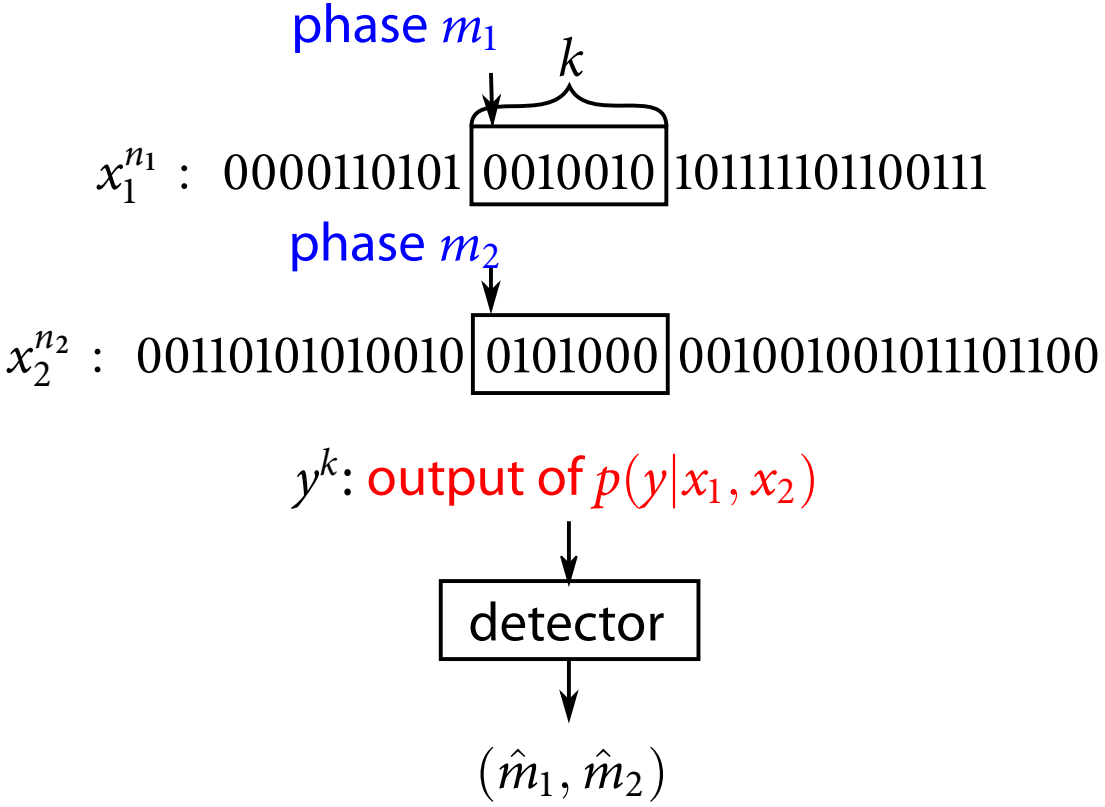
Problem setup



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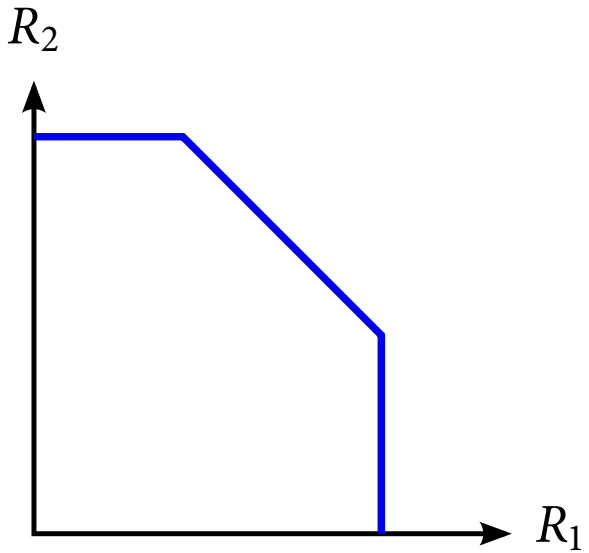

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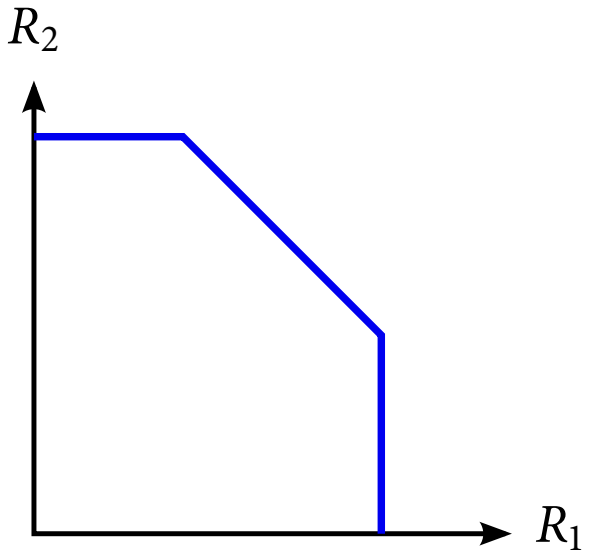
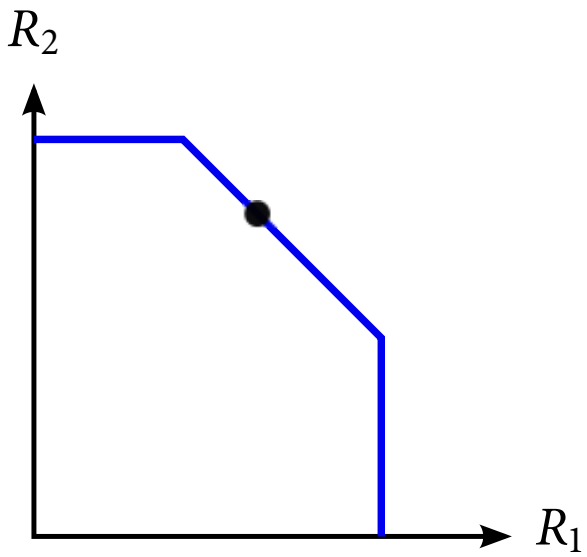


- Remarks
  - Strictly smaller than MAC capacity region !
  - Non-convex region (lack of synchronization)

A low-complexity construction

- Achieves arbitrary point in $\mathscr{C}_{\text{prob}}$
- $(k \log k)$ complexity



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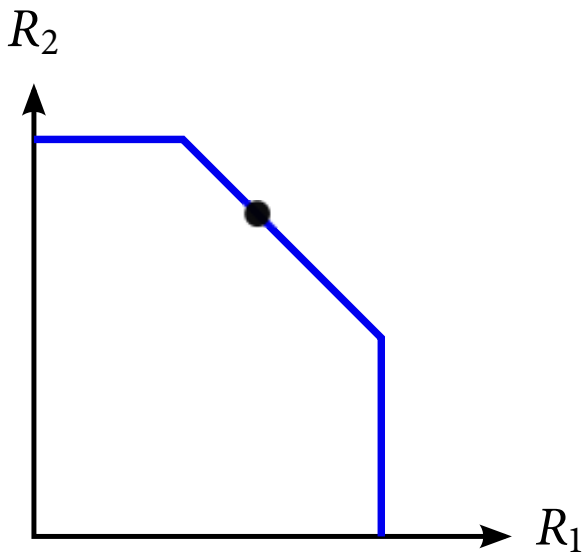

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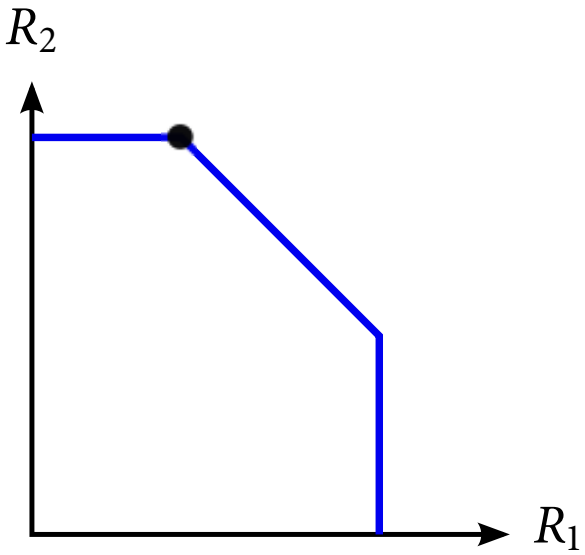

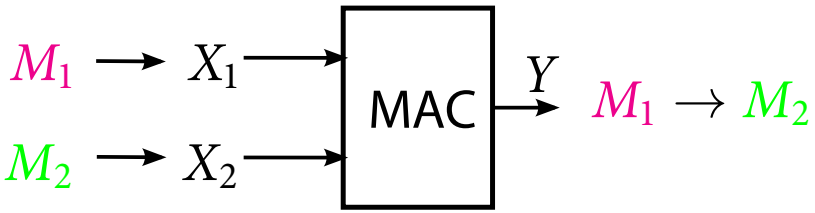

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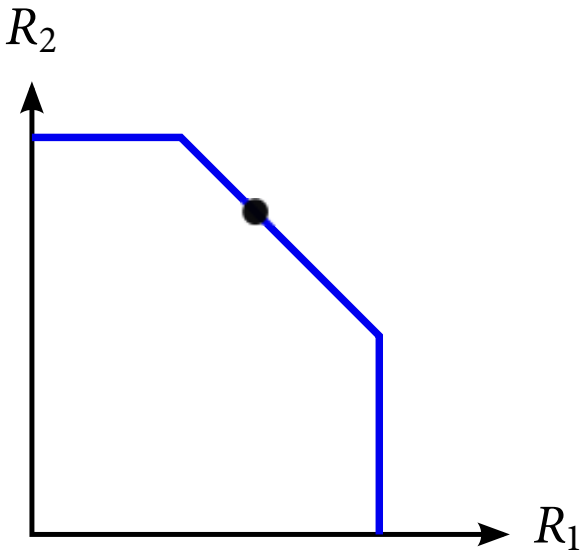

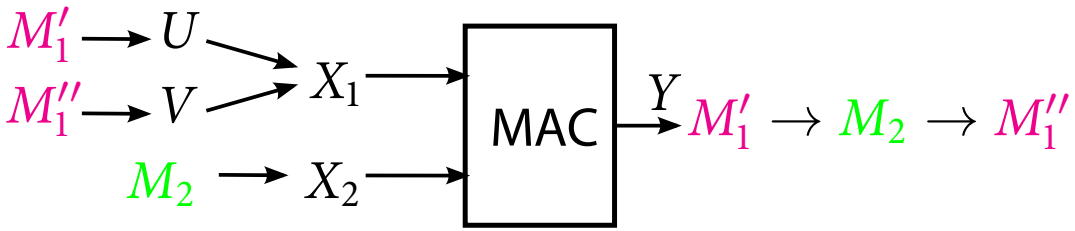

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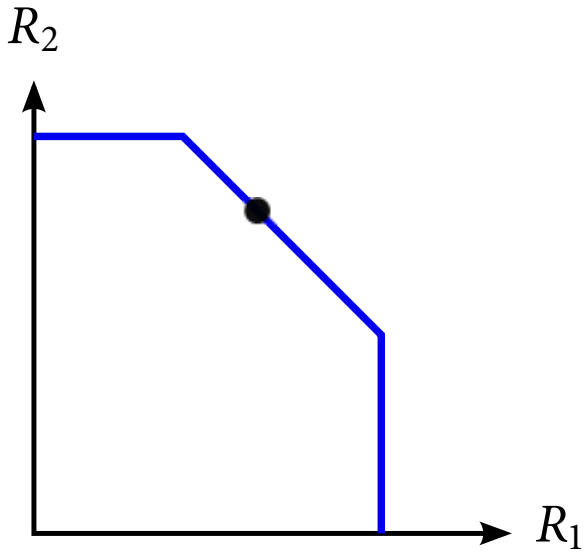

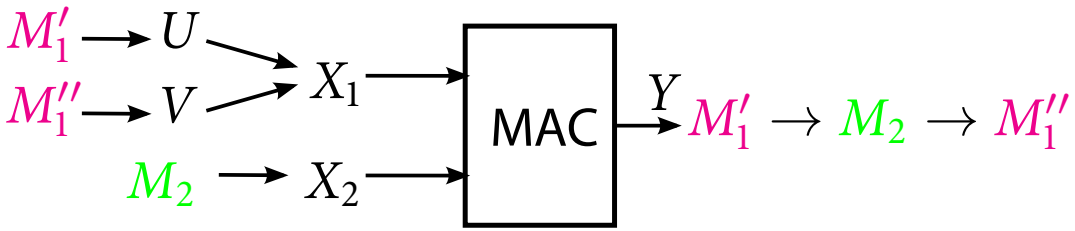

  - Symbol-by-symbol mapping $x_{1m} = f(u_i, v_j)$ with $i = m \pmod{n'}, j = m \pmod{n''}, gcd(n', n'') = 1$

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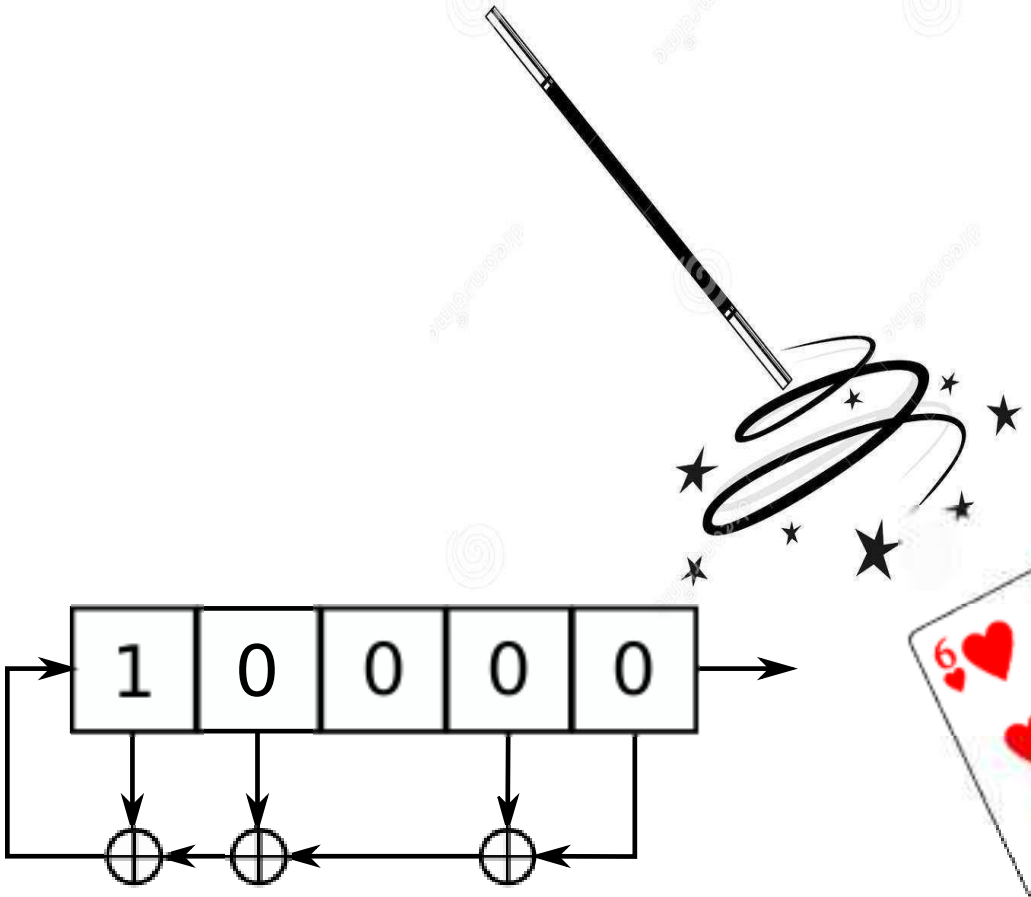
- Can we obtain tighter upper bound than coding theory?
- Adversarial phase detection for binary adder MAC: $Y = X_1 + X_2$?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