Vector Network Coding Based on Subspace Codes Outperforms Scalar Linear Network Coding

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*Technion CS Coding Seminar*
Outline

1 Motivation

2 The Combination Network

3 A Modified Combination Network

4 Another Modified Combination Network

5 Conclusion and Outlook
1 Motivation

2 The Combination Network

3 A Modified Combination Network

4 Another Modified Combination Network

5 Conclusion and Outlook
Routing vs. Network Coding

Routing:
- nodes simply forward packets

Network coding:
- nodes perform linear combinations

Network coding achieves higher throughput than routing!

Task in network coding: determine coefficients of linear combinations such that all receivers obtain all packets. (Here: no error-correction!)
Scalar and Vector Network Coding

For equivalent field sizes ($q_s = q^t$), vector network coding offers more freedom in choosing the coefficients!

- Scalar network coding:
  - Coefficients are scalars over field of size $q_s$.
  - $\sim$ For each coefficient: $q_s$ possibilities.

- Vector network coding of dimension $t$:
  - Coefficients are $t \times t$ matrices over field of size $q$.
  - $\sim$ For each coefficient: $q^{t^2}$ possibilities.

- [Sun et al., ISIT-2015]: There exist networks with a vector solution of dimension $t$ over $\mathbb{F}_q$, but no scalar solution for $q_s \leq q^t$.
- Our contribution: gap of size $q^{t^2/2} - q^t$ (and more)
Scalar and Vector Network Coding

\[ y = Ax_1 + Bx_2 \]

- **Scalar network coding:**
  - Coefficients are scalars over field of size \( q_s \)
  - \( \Rightarrow \) for each coefficient: \( q_s \) possibilities

- **Vector network coding of dimension \( t \):**
  - Coefficients are \( t \times t \) matrices over field of size \( q \)
  - \( \Rightarrow \) for each coefficient: \( q^{t^2} \) possibilities

For equivalent field sizes (\( q_s = q^t \)), vector network coding offers more freedom in choosing the coefficients!

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Scalar and Vector Network Coding

\[ y = Ax_1 + Bx_2 \]

- **Scalar network coding:**
  coefficients are *scalars* over field of size \( q_s \)
  \( \rightsquigarrow \) for each coefficient: \( q_s \) possibilities

- **Vector network coding of dimension \( t \):**
  coefficients are \( t \times t \) matrices over field of size \( q \)
  \( \rightsquigarrow \) for each coefficient: \( q^{t^2} \) possibilities

For equivalent field sizes (\( q_s = q^t \)), vector network coding offers more freedom in choosing the coefficients!

- [Sun *et al.*, ISIT-2015]: There exist networks with a vector solution of dimension \( t \) over \( \mathbb{F}_q \), but no scalar solution for \( q_s \leq q^t \).
- **Our contribution:** gap of size \( q^{t^2/2} - q^t \) (and more)
## Previous Work on Vector Network Coding

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5. Conclusion and Outlook
The Combination Network

\( \mathcal{N}_{h,r,s} \)-combination network 
\((s \geq h)\):

Three layers:
1. source with \( h \) messages
2. receive and forward linear combinations
3. \( \binom{r}{s} \) receivers: connected to \( s \) middle nodes, request all \( h \) messages

Theorem (Scalar Solution, e.g. [Riis & Ahlswede, 2006])

The \( \mathcal{N}_{h,r,s} \)-combination network has a scalar solution of field size \( q_s \) if and only if an \([r, h, d = r - s + 1]_{q_s}\) code exists.

\[ \implies \text{if } h = s: \text{ MDS code needed: } q_s \geq r - 1 \]
\( (\text{if } k \in \{3, q - 1\} \text{ } \& \text{ } q_s \text{ power of 2: } q_s \geq r - 2) \)
\[ \implies \text{decoding complexity: } \min\{\mathcal{O}(r \log^2 r), \mathcal{O}(h^{2.37})\} \text{ in } \mathbb{F}_{q_s} \]
The Combination Network

\[ \mathcal{N}_{h,r,s} \text{-combination network} \]
\((s \geq h)\):

Three layers:
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\]
Vector Network Coding in the Combination Network

**Construction**

- $\mathcal{D}_t = \{\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_{q^t}\}$: commutative MRD $[t \times t, t]_q$ code defined by the powers of a companion matrix $\mathbf{C}$
- consider $\mathcal{N}_{h,r,h}$-combination network with $r \leq q^t + 1$
- messages $\mathbf{x}_1, \ldots, \mathbf{x}_h$

One node from the middle layer receives and transmits $\mathbf{y}_r = \mathbf{x}_h$ and the other $r - 1$ nodes of the middle layer receive and transmit

$$\mathbf{y}_i = (\mathbf{I}_t \ \mathbf{C}_i \ \mathbf{C}_i^2 \ \ldots \ \mathbf{C}_i^{h-1}) \cdot (\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \ \mathbf{x}_h)^T \in \mathbb{F}_q^t, \ i = 1, \ldots, r - 1.$$

Each receiver obtains

$$
\begin{pmatrix}
\mathbf{y}_{i_1} \\
\vdots \\
\mathbf{y}_{i_{k-1}} \\
\mathbf{y}_{i_k}
\end{pmatrix}
= 
\begin{pmatrix}
\mathbf{I}_t & \mathbf{C}_{i_1} & \mathbf{C}_{i_1}^2 & \ldots & \mathbf{C}_{i_1}^{k-1} \\
\mathbf{I}_t & \mathbf{C}_{i_2} & \mathbf{C}_{i_2}^2 & \ldots & \mathbf{C}_{i_2}^{k-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{I}_t & \mathbf{C}_{i_k} & \mathbf{C}_{i_k}^2 & \ldots & \mathbf{C}_{i_k}^{k-1}
\end{pmatrix}
\begin{pmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2 \\
\vdots \\
\mathbf{x}_k
\end{pmatrix}
$$
Vector Network Coding in the Combination Network

Construction

- $D_t = \{C_1, C_2, \ldots, C_{q^t}\}$: commutative $\mathcal{MRD}[t \times t, t]^q$ code defined by the powers of a companion matrix $C$
- Consider $\mathcal{N}_{h,r,h}$-combination network with $r \leq q^t + 1$
- Messages $x_1, \ldots, x_h$

One node from the middle layer receives and transmits $y_r = x_h$ and the other $r - 1$ nodes of the middle layer receive and transmit $y_i = (I_t \ C_i \ C_i^2 \ \ldots \ C_i^{h-1}) \cdot (x_1 \ x_2 \ \ldots \ x_h)^T \in \mathbb{F}_q^t$, $i = 1, \ldots, r - 1$.

or

$$
\begin{pmatrix}
    y_{i_1} \\
    \vdots \\
    y_{i_{k-1}} \\
    y_1
\end{pmatrix} = 
\begin{pmatrix}
    I_t & C_{i_1} & C_{i_1}^2 & \ldots & C_{i_1}^{h-1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    I_t & C_{i_{h-1}} & C_{i_{h-1}}^2 & \ldots & C_{i_{h-1}}^{h-1} \\
    0_t & 0_t & 0_t & \ldots & I_t
\end{pmatrix} \cdot
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_k
\end{pmatrix}
$$
**Construction**

- \( \mathcal{D}_t = \{ \mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_{q^t} \} \): commutative \( \mathcal{M} \mathcal{R} \mathcal{D}[t \times t, t]_q \) code defined by the powers of a companion matrix \( \mathbf{C} \)

- Consider \( \mathcal{N}_{h,r,h} \)-combination network with \( r \leq q^t + 1 \)

- Messages \( \mathbf{x}_1, \ldots, \mathbf{x}_h \)

One node from the middle layer receives and transmits \( \mathbf{y}_r = \mathbf{x}_h \) and the other \( r - 1 \) nodes of the middle layer receive and transmit

\[
\mathbf{y}_i = \left( \mathbf{I}_t \mathbf{C}_i \mathbf{C}_i^2 \ldots \mathbf{C}_i^{h-1} \right) \cdot \left( \mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_h \right)^T \in \mathbb{F}_q^t, \ i = 1, \ldots, r - 1.
\]

or

\[
\begin{pmatrix}
\mathbf{y}_{i_1} \\
\vdots \\
\mathbf{y}_{i_{k-1}} \\
\mathbf{y}_1
\end{pmatrix} =
\begin{pmatrix}
\mathbf{I}_t & \mathbf{C}_{i_1} & \mathbf{C}_{i_1}^2 & \ldots & \mathbf{C}_{i_1}^{h-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{I}_t & \mathbf{C}_{i_{h-1}} & \mathbf{C}_{i_{h-1}}^2 & \ldots & \mathbf{C}_{i_{h-1}}^{h-1} \\
0_t & 0_t & 0_t & \ldots & \mathbf{I}_t
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2 \\
\vdots \\
\mathbf{x}_k
\end{pmatrix}
\]

\( \implies \) matrices have full rank (due to commutative \( \mathcal{M} \mathcal{R} \mathcal{D} \) code)

\( \implies \) there is a unique solution for \( (\mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_h) \)
Theorem

For the $\mathcal{N}_{h,q^t+1,h}$-combination network, a vector linear solution of field size $q$ and dimension $t$ exists.

The decoding complexity is in $\mathcal{O}(t^2 h \log^2 h)$ over $\mathbb{F}_q$ for each receiver.

Comparison to scalar solution:

- equivalent field sizes ($q^t$)
- reduction of decoding complexity:
  for small $h$ improvement from $h^{2.34}$ to quasi-linear in $h$

Remark: $k = 2$ was solved in [Sun et al., ISIT-2015].
Theorem

For the $N_{h,q^t+1,h}$-combination network, a vector linear solution of field size $q$ and dimension $t$ exists.

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A Modified Combination Network

\[ \mathcal{N}^*_{4,r,4} \text{ network:} \]

Three layers:
1. Source with \( h = 4 \) messages
2. Receive and forward linear combinations
3. \( \binom{r}{2} \) receivers: connected to 2 middle nodes, request all 4 messages, extra link from source

Theorem (Scalar Solution)

There is a scalar linear solution of field size \( q_s \) for the \( \mathcal{N}^*_{4,r,4} \)-network if and only if \( (q_s^2 + 1)(q_s^2 + q_s + 1) \geq r \).

Proof: next slide.

\[ q_s \in O\left(r^{1/4}\right) \]
A Modified Combination Network

$\mathcal{N}_{4,r,4}^*$ network:

Three layers:

1. source with $h = 4$ messages
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Proof: next slide.

$\implies q_s \in \mathcal{O}(r^{1/4})$
A Modified Combination Network—Scalar Solution

Theorem (Scalar Solution)
There is a scalar linear solution of field size $q_s$ for the $N_{4,r,4}^*$-network if and only if $(q_s^2 + 1)(q_s^2 + q_s + 1) \geq r$.

Proof idea:

- $B$ is $4 \times 2r$-matrix divided into $r$ blocks of two columns such that any two blocks together have rank at least three.
- Transmit symbols of one block of $(x_1, x_2, x_3, x_4) \cdot B$ at each middle node.
- Extra links: transmit symbol $p_i$ such that full rank at receiver.
- Blocks: $4 \times 2$-matrix representations of all 2-dimensional subspaces of $\mathbb{F}_{q_s}^4$.

\[ r \leq \binom{4}{2} q_s = (q_s^2 + 1)(q_s^2 + q_s + 1) \in \mathcal{O}(q_s^4) \]

- "only if" can be shown since there is no scheme that provides more blocks.

\[ \implies q_s \in \mathcal{O}(r^{1/4}) \]
Let $\mathcal{C} = \{C_1, C_2, \ldots, C_{q^{2t^2+2t}}\}$ be an $\mathcal{MRD}[2t \times 2t, t]_q$ code and let $r \leq q^{2t^2+2t}$. Consider the $\mathcal{N}^*_4,r,4$-network with message vectors $x_1, x_2, x_3, x_4 \in \mathbb{F}_q^t$. The $i$-th middle node receives and transmits:

\[
\begin{pmatrix}
  y_{i_1} \\
  y_{i_2}
\end{pmatrix} = \begin{pmatrix} I_{2t} & C_i \end{pmatrix} \cdot \begin{pmatrix} x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix} \in \mathbb{F}_q^{2t}, \quad i = 1, \ldots, r.
\]

The extra link from the source which ends in the same receiver as the links from two distinct nodes $i, j \in \{1, 2, \ldots, r\}$ of the middle layer transmits the vector $z_{ij} = P_{ij} \cdot (x_1, x_2, x_3, x_4)^T \in \mathbb{F}_q^t$, where the $t \times 4t$ matrix $P_{ij}$ is chosen such that

\[
\text{rk} \begin{pmatrix} I_{2t} & C_i \\
  I_{2t} & C_j \\
  P_{ij}
\end{pmatrix} = 4t.
\]
Our construction provides a vector solution of field size $q$ and dimension $t$ to the $N_{4,r,4}^*$-network for $r \leq q^{2t(t+1)}$.

**Proof:** On each receiver, we obtain

$$
\begin{pmatrix}
  y_{i_1} \\
  y_{i_2} \\
  y_{j_1} \\
  y_{j_2} \\
  z_{ij}
\end{pmatrix} = 
\begin{pmatrix}
  I_{2t} & C_i \\
  I_{2t} & C_j \\
  0 & P_{ij}
\end{pmatrix} \cdot 
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix}.
$$

Since $\text{rk} \left( \begin{pmatrix} I_{2t} & C_i \\ I_{2t} & C_j \end{pmatrix} \right) \geq 3t$, $P_{ij}$ can be chosen such that overall rank is $4t$. $
\implies$ unique solution for $(x_1, x_2, x_3, x_4)$

Comparison to scalar solution: $q_s \in O(q^{t^2/2})$ vs. $q^t$

$\implies$ significant reduction of field size!
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**Comparison to scalar solution:** $q_s \in O(q^{t^2/2})$ vs. $q^t$

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**Generalization**

$N_{h,r,h}^*$-combination network with $\ell$ parallel edges ($h = 2\ell$)

Three layers:

1. source with $h = 2\ell$ ($\ell \geq 2$) messages
2. receive and forward linear combinations, $\ell$ parallel edges
3. $\binom{r}{2}$ receivers: connected to $\ell$ middle nodes, request all $h$ messages, gets $2\ell + 1$ links

**Solutions**

- scalar solution exists if and only if $r \leq O(q_s^{2\ell})$
- vector solution: use $MRD[\ell t \times \ell t, (\ell - 1)t]_q$ of size $q^{\ell t^2 + \ell t}$

**Comparison:** $q_s \in O(q_t^{t^2/2})$ vs. $q^t$ (for any $k = 2\ell$, $\ell \geq 2$)

$\longrightarrow$ significant reduction of field size!
Generalization

\( N_{h,r,h}^{\ast} \)-combination network with \( \ell \) parallel edges (\( h = 2\ell \))

Three layers:
1. source with \( h = 2\ell \) (\( \ell \geq 2 \)) messages
2. receive and forward linear combinations, \( \ell \) parallel edges
3. \( \binom{r}{2} \) receivers: connected to \( \ell \) middle nodes, request all \( h \) messages, gets \( 2\ell + 1 \) links

Solutions
- scalar solution exists if and only if \( r \leq O(q_{s}^{2\ell}) \)
- vector solution: use \( \mathcal{MRD}[\ell t \times \ell t, (\ell - 1)t]_{q} \) of size \( q^{\ell t^{2} + \ell t} \)

Comparison: \( q_{s} \in O(q^{t^{2}/2}) \) vs. \( q^{t} \) (for any \( k = 2\ell, \ell \geq 2 \))

\( \Rightarrow \) significant reduction of field size!
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Another Modified Combination Network

$N_{h,r,h}^+$-combination network with $\ell$ parallel edges ($h = 2\ell$) and $\ell - 1$ direct links

Three layers:

1. source with $h = 2\ell$ ($\ell \geq 2$) messages
2. receive and forward linear combinations, $\ell$ parallel edges
3. $\binom{r}{2}$ receivers: connected to $\ell$ middle nodes, request all $h$ messages, gets $3\ell - 1$ links

Lemma (Scalar Solution)

There is a scalar solution of field size $q_s$ for the $N_{h,r,h}^+$-network with $h = 2\ell$ messages, where $\ell \geq 2$, if and only if $q_s^{\ell^2} \geq r$.

$\implies q_s \in O\left(\frac{r}{\ell^2}\right)$
Another Modified Combination Network

\( \mathcal{N}_{h,r,h}^+ \)-combination network

with \( \ell \) parallel edges \((h = 2\ell)\)

and \( \ell - 1 \) direct links

Three layers:
1. source with \( h = 2\ell \) \((\ell \geq 2)\) messages
2. receive and forward linear combinations, \( \ell \) parallel edges
3. \( \binom{r}{2} \) receivers: connected to \( \ell \) middle nodes, request all \( h \) messages, gets \( 3\ell - 1 \) links

Lemma (Scalar Solution)

There is a scalar solution of field size \( q_s \) for the \( \mathcal{N}_{h,r,h}^+ \)-network

with \( h = 2\ell \) messages, where \( \ell \geq 2 \), if and only if \( q_s^{\ell^2} \geq r \).

\[ \implies q_s \in \mathcal{O}(r^{1/\ell^2}) \]
Vector Network Coding

Construction

\{C_1, \ldots, C_{q^{\ell (\ell -1) t^2 + \ell t}}\} is $\mathcal{MRD}[\ell t \times \ell t, t]_q$ code and $r \leq q^{\ell (\ell -1) t^2 + \ell t}$.

Consider the $\mathcal{N}^+_{h,r,h}$-network with messages $x_1, \ldots, x_h \in \mathbb{F}_q^t$, where $h = 2\ell$, $\ell \geq 2$. The $i$-th middle node receives and transmits:

$$
\begin{pmatrix}
y_{i1} \\
y_{i2}
\end{pmatrix} = (I_{\ell t} \ C_i) \cdot \begin{pmatrix}x_1 \\
\vdots \\
x_{2\ell}
\end{pmatrix} \in \mathbb{F}_q^{\ell t}, \quad i = 1, \ldots, r.
$$

The $\ell - 1$ direct links from the source transmit the vectors $z_{ijs} = P_{ijs} \cdot (x_1, \ldots, x_{2\ell})^T \in \mathbb{F}_q^t$, for $s = 1, \ldots, \ell - 1$, where the $t \times 2\ell t$ matrices $P_{ijs}$ are chosen s.t.

$$
\text{rk} \begin{pmatrix}I_{\ell t} & C_i \\
I_{\ell t} & C_j \\
\vdots \\
P_{ij1} \\
P_{ij(\ell -1)}
\end{pmatrix} = 2\ell t.
$$
Our construction provides a vector solution of field size $q$ and dimension $t$ to the $N^+_{h,r,h}$-network for $r \leq q^{\ell(\ell-1)t^2 + \ell t}$, where $h = 2\ell$ and $\ell \geq 2$.

Proof:
Since $\text{rk} \left( \begin{pmatrix} I_{\ell t} & C_i \\ I_{\ell t} & C_j \end{pmatrix} \right) \geq \ell t + t = (\ell + 1)t$, the $(\ell - 1)t$ rows of the matrices $P_{ijs}$ can be chosen such that the overall rank $2\ell t$. \[\implies\] unique solution for $(x_1, \ldots, x_{2\ell})$

Comparison to scalar solution:
$q_s \in \mathcal{O}(r^{1/\ell^2}) = \mathcal{O}(q^{(1-\frac{1}{\ell})t^2})$ vs. $q^t$

\[\implies\] significant reduction of field size!
Vector Network Coding

**Theorem**

Our construction provides a vector solution of field size \( q \) and dimension \( t \) to the \( \mathcal{N}_{h,r,h}^\dagger \)-network for \( r \leq q^{\ell(\ell-1)t^2+\ell t} \), where \( h = 2\ell \) and \( \ell \geq 2 \).

**Proof:**

Since \( \text{rk} \left( \begin{bmatrix} I_{\ell t} & C_i \\ I_{\ell t} & C_j \end{bmatrix} \right) \geq \ell t + t = (\ell + 1)t \), the \( (\ell - 1)t \) rows of the matrices \( P_{ijs} \) can be chosen such that the overall rank \( 2\ell t \).

\[ \implies \text{unique solution for } (x_1, \ldots, x_{2\ell}) \]

**Comparison to scalar solution:**

\[ q_s \in \mathcal{O}(r^{1/\ell^2}) = \mathcal{O}(q^{(1-\frac{1}{\ell})t^2}) \text{ vs. } q^t \]

\[ \implies \text{significant reduction of field size!} \]
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Vector Network Coding

Contributions:

- Use **MRD matrices** as vector network coding coefficients
  \[ \Rightarrow \text{reduce decoding complexity} \]
  \[ \Rightarrow \text{large gap in field size of scalar \& vector coding:} \]
  \[ q^{(1 - \frac{1}{\ell})t^2} - q^t \text{ for any } \ell \geq 2 \]
- Slight improvement by using other subspace codes
- For \( h = 3 \): gap of smaller size for some parameters

Open Questions:

- Networks with \( h = 2, 3 \) and large gap?
- Largest possible gap?
- Network with \( h \) edge disjoint paths and large gap?
- Gap between non-linear and linear network coding?
Contributions:

- Use **MRD matrices** as vector network coding coefficients
  - $\Rightarrow$ reduce decoding complexity
  - $\Rightarrow$ large gap in field size of scalar & vector coding:
    \[ q^{(1-\frac{1}{t})t^2} - q^t \text{ for any } \ell \geq 2 \]
- Slight improvement by using other subspace codes
- For $h = 3$: gap of smaller size for some parameters

Open Questions:

- Networks with $h = 2, 3$ and large gap?
- Largest possible gap?
- Network with $h$ edge disjoint paths and large gap?
- Gap between non-linear and linear network coding?
Thank you...

...for your attention!

Questions?