Consecutive Switch Codes for Network Switches

Sarit Buzaglo[1], Eitan Yaakobi[2], Yuval Cassuto[2], and Paul H. Siegel[1]

[1]CMRR

[2]Technion

Coding Theory Seminar
July 3, 2016

Outline

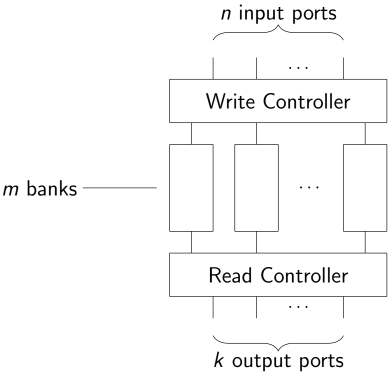
- Introduction
- Construction of binary switch codes.
- Consecutive switch codes.
- Conclusion

Network switches

A **network switch** consists of n input ports, k output ports, and m **banks**.

In each time slot, called **generation** n input packets are processed and stored in the banks.

A read controller outputs any request of k packets by accessing at most one packet from every bank.



Switch codes

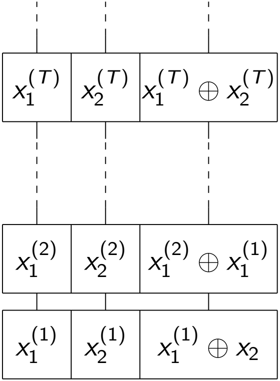
Switch codes are coding schemes that are designed to read and write data in network switches. A **$(n,k,m)_q$-switch code** is defined as follows.

1) For every $T \geq 1$, a vector $\mathbf{x}^{(T)} \in \mathbb{F}_q^n$ is encoded to a vector $\mathbf{c}^{(T)} \in \mathbb{F}_q^m$.
2) For every **request set** of k packets $I = \{x_{i_1}^{(T_1)}, x_{i_2}^{(T_2)}, \ldots, x_{i_k}^{(T_k)}\}$, there exist k disjoint **recovery sets** $J_1, J_2, \ldots, J_k \subset [m]$ such that for every $1 \leq r \leq k$, the packet $x_{i_r}^{(T_r)}$ can be recovered from $\{c_j^{(T_r)}\}_{j \in J_r}$. We call $\{c_j^{(T_r)}\}_{j \in J_r}$ the **recovery subsequence** for $x_{i_r}^{(T_r)}$.

Switch codes

Example

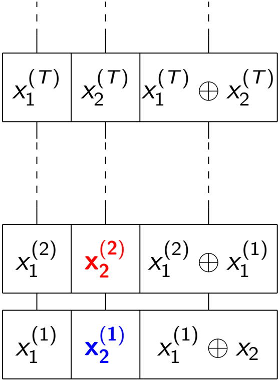
A $(n = 2, k = 2, m = 3)$ binary switch code



Switch codes

Example

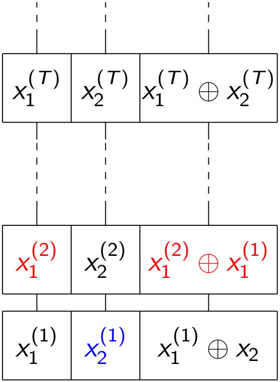
A $(n = 2, k = 2, m = 3)$ binary switch code



Switch codes

Example

A $(n = 2, k = 2, m = 3)$ binary switch code



Switch codes

In this talk we consider only the case $n = k$ and refer to switch codes as $(n, m)_q$-**switch codes**.
In particular we are interested in binary codes.

Switch codes

In this talk we consider only the case $n = k$ and refer to switch codes as $(n, m)_q$-**switch codes**.
In particular we are interested in binary codes.

Related work

- Wang et al., 2013- Presented switch codes, constructions, and bounds in terms of the degree.
- Wang et al., 2015- Construction of binary switch code with $m = n^2/\log_2 n$.
- Ishai et al., 2004- Presented and constructed batch codes. A construction of a binary switch code with $m = n^{\log_2 3} > n^{1.5849}$.

Switch codes

In this talk we consider only the case $n = k$ and refer to switch codes as $(n, m)_q$-**switch codes**.
In particular we are interested in binary codes.

Related work

- Wang et al., 2013- Presented switch codes, constructions, and bounds in terms of the degree.
- Wang et al., 2015- Construction of binary switch code with $m = n^2 / \log_2 n$.
- Ishai et al., 2004- Presented and constructed batch codes. A construction of a binary switch code with $m = n^{\log_2 3} > n^{1.5849}$.

We will show a construction of binary switch code with $m \approx 2n^{1.5}$.

Construction of switch code

A **(ν, n)-one-step majority code with availability s** is a code, $\mathcal{C}$, that encodes $\mathbf{x} \in \mathbb{F}_q^n$ to $\mathcal{E}(\mathbf{x}) \in \mathbb{F}_q^{(\nu)}$ and for all $i \in [n]$, there exist s disjoint subsets $J_1, J_2, \ldots, J_s \subset [\nu]$, such that for every $1 \leq r \leq s$, the packet x_i can be recovered from $\{\mathcal{E}(\mathbf{x})_j\}_{j \in J_r}$.

Construction of switch code

A **(ν, n)-one-step majority code with availability s** is a code, $\mathcal{C}$, that encodes $\mathbf{x} \in \mathbb{F}_q^n$ to $\mathcal{E}(\mathbf{x}) \in \mathbb{F}_q^{(\nu)}$ and for all $i \in [n]$, there exist s disjoint subsets $J_1, J_2, \ldots, J_s \subset [\nu]$, such that for every $1 \leq r \leq s$, the packet x_i can be recovered from $\{\mathcal{E}(\mathbf{x})_j\}_{j \in J_r}$.

Theorem

If $\mathcal{C}$ is a (ν, n)-one-step majority code over $\mathbb{F}_q$ with availability s, where $s \leq n$, then the code that maps $\mathbf{x}$ to

$$\mathbf{c} = \underbrace{\mathbf{x}|\mathbf{x}|\cdots|\mathbf{x}}_{s \text{ times}} \,|\, \underbrace{\mathcal{E}(\mathbf{x})|\mathcal{E}(\mathbf{x})|\cdots|\mathcal{E}(\mathbf{x})}_{\lfloor n/s \rfloor \text{ times}}$$

is a $(n, m)_q$-switch code, with $m = sn + \lfloor n/s \rfloor \nu$.

Proof.

- It is suffice to show that for any multi-set $I = \{i_1, i_2, \ldots, i_n\}$ of n indices, the information symbols $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ can be recovered from n disjoint recovery subsequences of $\mathbf{c}$.

Proof.

- It is suffice to show that for any multi-set $I = \{i_1, i_2, \ldots, i_n\}$ of n indices, the information symbols $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ can be recovered from n disjoint recovery subsequences of $\mathbf{c}$.
- For every $1 \le i \le n$, let r_i be the number of appearances of i in the multi-set I and let $m_i = \min\{s, r_i\}$. Hence,

$$\sum_{i=1}^{n} r_i = n.$$

Proof.

- It is suffice to show that for any multi-set $I = \{i_1, i_2, \ldots, i_n\}$ of n indices, the information symbols $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ can be recovered from n disjoint recovery subsequences of $\mathbf{c}$.
- For every $1 \leq i \leq n$, let r_i be the number of appearances of i in the multi-set I and let $m_i = \min\{s, r_i\}$. Hence,

$$\sum_{i=1}^{n} r_i = n.$$

- Since $m_i \leq s$, we can recover m_i copies of x_i directly from the s copies of $\mathbf{x}$.

Proof Continues

- To recover the remaining $r_i - m_i$ copies of x_i we first find s disjoint subsequences of $\mathbf{z} = \mathcal{E}(\mathbf{x})$ that can each recover x_i. Then, we use at most $\left\lceil \frac{r_i - m_i}{s} \right\rceil \leq \frac{r_i}{s}$ copies of each of the s subsequences to recover x_i.

Proof Continues

- To recover the remaining $r_i - m_i$ copies of x_i we first find s disjoint subsequences of $\mathbf{z} = \mathcal{E}(\mathbf{x})$ that can each recover x_i. Then, we use at most $\left\lceil \frac{r_i - m_i}{s} \right\rceil \leq \frac{r_i}{s}$ copies of each of the s subsequences to recover x_i.
- For every $1 \leq j \leq \nu$ and for every $1 \leq i \leq n$ we used at most $\frac{r_i}{s}$ copies of z_j to recover x_i. Hence, we used at most

$$\left\lfloor \sum_{i=1}^{n} \frac{r_i}{s} \right\rfloor \leq \left\lfloor \frac{n}{s} \right\rfloor$$

copies of z_j and indeed $\mathbf{c}$ consists of enough copies of $\mathbf{z}$.

Binary switch codes

Lemma (Lin and Costello, 2004, p. 293)

The binary cyclic $(\nu = 2^{2r} + 2^r + 1, n = 2^{2r} + 2^r - 3^r)$*-difference-set code is a binary* (ν, n)*-one-step majority code with availability* $s = 2^r + 1 \approx \sqrt{n}$*.*

Binary switch codes

Lemma (Lin and Costello, 2004, p. 293)

The binary cyclic $(\nu = 2^{2r} + 2^r + 1, n = 2^{2r} + 2^r - 3^r)$*-difference-set code is a binary* (ν, n)*-one-step majority code with availability* $s = 2^r + 1 \approx \sqrt{n}$*.*

Corollary

There exists an (explicit) $(n, m)_2$*-switch code, with*

$$m = sn + \left\lfloor \frac{n}{s} \right\rfloor \nu \approx 2n^{1.5}.$$

Consecutive switch codes

A ℓ-**consecutive switch code** is a switch code in which the request sets are restricted to ℓ consecutive generations, i.e., of the form

$$I = \{x_{i_1}^{(T_1)}, x_{i_2}^{(T_2)}, \ldots, x_{i_k}^{(T_k)}\}$$

and

$$T_1, T_2, \ldots, T_k \in \{T, T+1, T+2, \ldots, T+\ell-1\}.$$

Consecutive switch codes

A **ℓ-consecutive switch code** is a switch code in which the request sets are restricted to ℓ consecutive generations, i.e., of the form

$$I = \{x_{i_1}^{(T_1)}, x_{i_2}^{(T_2)}, \ldots, x_{i_k}^{(T_k)}\}$$

and

$$T_1, T_2, \ldots, T_k \in \{T, T+1, T+2, \ldots, T+\ell-1\}.$$

Advantage: Admit better rates (n/m) and yet deliver a large collection of common request sets.

Consecutive switch codes

Example

A $(n = 4, k = 4, m = 6)_2$-ℓ-consecutive switch code with $\ell = 2$.

$x_1^{(2)}$	$x_2^{(2)}$	$x_3^{(2)}$	$x_4^{(2)}$	$x_3^{(2)} \oplus x_4^{(2)}$	$x_1^{(2)} \oplus x_2^{(2)}$
$x_1^{(1)}$	$x_2^{(1)}$	$x_3^{(1)}$	$x_4^{(1)}$	$x_1^{(1)} \oplus x_2^{(1)}$	$x_3^{(1)} \oplus x_4^{(1)}$

Consecutive switch codes

Example

A $(n = 4, k = 4, m = 6)_2$-ℓ-consecutive switch code with $\ell = 2$.

$\mathbf{x_1^{(2)}}$	$\mathbf{x_2^{(2)}}$	$x_3^{(2)}$	$x_4^{(2)}$	$x_3^{(2)} \oplus x_4^{(2)}$	$x_1^{(2)} \oplus x_2^{(2)}$
$\mathbf{x_1^{(1)}}$	$\mathbf{x_2^{(1)}}$	$x_3^{(1)}$	$x_4^{(1)}$	$x_1^{(1)} \oplus x_2^{(1)}$	$x_3^{(1)} \oplus x_4^{(1)}$

Consecutive switch codes

Example

A $(n=4, k=4, m=6)_2$-ℓ-consecutive switch code with $\ell = 2$.

$x_1^{(2)}$	$x_2^{(2)}$	$x_3^{(2)}$	$x_4^{(2)}$	$x_3^{(2)} \oplus x_4^{(2)}$	$x_1^{(2)} \oplus x_2^{(2)}$
$x_1^{(1)}$	$x_2^{(1)}$	$x_3^{(1)}$	$x_4^{(1)}$	$x_1^{(1)} \oplus x_2^{(1)}$	$x_3^{(1)} \oplus x_4^{(1)}$

Combinatorial switch code

In a **combinatorial** switch code, the entries of a codeword $\mathbf{c}^T$ are simply copies of the entries of the input vector $\mathbf{x}^T$.

A combinatorial ℓ-consecutive switch code can be represented by a matrix $F \in [n]^{\ell \times m}$.

Combinatorial switch code

In a **combinatorial** switch code, the entries of a codeword $\mathbf{c}^T$ are simply copies of the entries of the input vector $\mathbf{x}^T$.

A combinatorial ℓ-consecutive switch code can be represented by a matrix $F \in [n]^{\ell \times m}$.

Example

A combinatorial $(n = 3, k = 3, m = 5)$-2-consecutive switch code

$x_1^{(2)}$	$x_2^{(2)}$	$x_3^{(2)}$	$x_3^{(2)}$	$x_3^{(2)}$
$x_1^{(1)}$	$x_2^{(1)}$	$x_3^{(1)}$	$x_1^{(1)}$	$x_2^{(1)}$

$$F = \begin{pmatrix} 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 \end{pmatrix} \in [3]^{2\times 5}$$

Combinatorial switch codes for $\ell = 2$

Henceforth, we will focus only on combinatorial consecutive switch codes and the case $k = n$. We refer to combinatorial consecutive switch codes as **(n, m)-ℓ-switch codes**.

Combinatorial switch codes for $\ell = 2$

Henceforth, we will focus only on combinatorial consecutive switch codes and the case $k = n$. We refer to combinatorial consecutive switch codes as **(n, m)-ℓ-switch codes**.

Theorem

For every (n, m)-2-switch code we have $m \geq 2n - 1$.

Combinatorial switch codes for $\ell = 2$

Henceforth, we will focus only on combinatorial consecutive switch codes and the case $k = n$. We refer to combinatorial consecutive switch codes as (n, m)-ℓ-**switch codes**.

Theorem

For every (n, m)-2-switch code we have $m \geq 2n - 1$.

Remark: The ℓ-repetition code, formed by repeating each input symbol ℓ-times, is a (n, m)-ℓ-switch code with $m = \ell n$.

Combinatorial switch codes for $\ell = 2$

Henceforth, we will focus only on combinatorial consecutive switch codes and the case $k = n$. We refer to combinatorial consecutive switch codes as (n, m)-ℓ-**switch codes**.

Theorem

For every (n, m)-2-switch code we have $m \geq 2n - 1$.

Remark: The ℓ-repetition code, formed by repeating each input symbol ℓ-times, is a (n, m)-ℓ-switch code with $m = \ell n$.

Q: Can we construct a (n, m)-2-switch code with $m = 2n - 1$?

Combinatorial switch codes for $\ell = 2$

Henceforth, we will focus only on combinatorial consecutive switch codes and the case $k = n$. We refer to combinatorial consecutive switch codes as (n, m)-ℓ-**switch codes**.

Theorem

For every (n, m)-2-switch code we have $m \geq 2n - 1$.

Remark: The ℓ-repetition code, formed by repeating each input symbol ℓ-times, is a (n, m)-ℓ-switch code with $m = \ell n$.

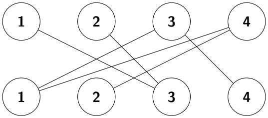
Q: Can we construct a (n, m)-2-switch code with $m = 2n - 1$? Yes.

$$F = \begin{pmatrix} 1 & 2 & \cdots & n & n & n & \cdots & n \\ 1 & 2 & \cdots & n & 1 & 2 & \cdots & n-1 \end{pmatrix}$$

Example

For $n = 4$ and $m = 6$:

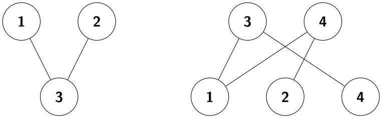
$$F = \begin{pmatrix} 4 & 3 & 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 3 & 1 & 2 \end{pmatrix}$$



Example

For $n = 4$ and $m = 6$:

$$F = \begin{pmatrix} 4 & 3 & 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 3 & 1 & 2 \end{pmatrix}$$



Construction of a (n, m)-3-switch code

We construct a combinatorial (n, m)-3-switch code of the following form

$$F = (F_1 \mid F_2 \mid F_3),$$

where

$$F_1 = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix},$$

$F_2 \in [n]^{3\times\nu}$, and $F_3 \in [n]^{3\times\nu}$ is obtained by a cyclic shift of the rows of F_2.

Construction of a (n, m)-3-switch code

We construct a combinatorial (n, m)-3-switch code of the following form

$$F = (F_1 \mid F_2 \mid F_3),$$

where

$$F_1 = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix},$$

$F_2 \in [n]^{3\times\nu}$, and $F_3 \in [n]^{3\times\nu}$ is obtained by a cyclic shift of the rows of F_2.

Lemma

If every $i_1, i_2, \ldots, i_s \in [n]$, $s \leq n/2$, can be found in s distinct columns of F_2 then F is a (n, m)-3-switch code with $m = n + 2\nu$.

Constructing F_2

Let $n = \alpha(\alpha+1)$. $F_2 \in [n]^{3\times\nu}$, $\nu = n - \alpha$, is of the form

$$F_2 = (A_1 \mid A_2 \mid \cdots \mid A_\alpha),$$

where

$$A_r = \begin{pmatrix} r\alpha+2 & r\alpha+3 & \cdots & (r+1)\alpha & r\alpha+1 \\ r\alpha+1 & r\alpha+2 & \cdots & r\alpha+\alpha-1 & (r+1)\alpha \\ 1 & 2 & \cdots & \alpha-1 & \alpha \end{pmatrix},$$

$1 \le r \le \alpha$.

Constructing F_2

Let $n = \alpha(\alpha+1)$. $F_2 \in [n]^{3\times\nu}$, $\nu = n - \alpha$, is of the form

$$F_2 = (A_1 \mid A_2 \mid \cdots \mid A_\alpha),$$

where

$$A_r = \begin{pmatrix} r\alpha+2 & r\alpha+3 & \cdots & (r+1)\alpha & r\alpha+1 \\ r\alpha+1 & r\alpha+2 & \cdots & r\alpha+\alpha-1 & (r+1)\alpha \\ 1 & 2 & \cdots & \alpha-1 & \alpha \end{pmatrix},$$

$1 \le r \le \alpha$.

Lemma

Every $i_1, i_2, \ldots, i_s \in [n]$, $s \le n/2$, can be found in s distinct columns of F_2

Constructing F_2

Let $n = \alpha(\alpha+1)$. $F_2 \in [n]^{3\times\nu}$, $\nu = n - \alpha$, is of the form

$$F_2 = (A_1 \mid A_2 \mid \cdots \mid A_\alpha),$$

where

$$A_r = \begin{pmatrix} r\alpha+2 & r\alpha+3 & \cdots & (r+1)\alpha & r\alpha+1 \\ r\alpha+1 & r\alpha+2 & \cdots & r\alpha+\alpha-1 & (r+1)\alpha \\ 1 & 2 & \cdots & \alpha-1 & \alpha \end{pmatrix},$$

$1 \le r \le \alpha$.

Lemma

Every $i_1, i_2, \ldots, i_s \in [n]$, $s \le n/2$, can be found in s distinct columns of F_2

Corollary

F is a (n, m)-3-switch code with $n = \alpha(\alpha+1)$ and $m = 3n - 2\alpha \approx 3n - 2\sqrt{n}$.

Combinatorial (n, m)-3-switch code

Example

Let $n = 6$, then $\alpha = 2$,

$$F_2 = (A_1|A_2) = \left(\begin{array}{cc|cc} 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 \end{array}\right),$$

and

$$F = (F_1|F_2|F_3) = \left(\begin{array}{cccccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 4 & 3 & 6 & 5 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 2 & 4 & 3 & 6 & 5 \end{array}\right).$$

Combinatorial (n, m)-3-switch code

Example

Let $n = 6$, then $\alpha = 2$,

$$F_2 = (A_1|A_2) = \left(\begin{array}{cc|cc} 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 \end{array}\right),$$

and

$$F = (F_1|F_2|F_3) = \left(\begin{array}{cccccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 4 & 3 & 6 & 5 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 2 & 4 & 3 & 6 & 5 \end{array}\right).$$

Combinatorial (n, m)-3-switch code

Example

Let $n = 6$, then $\alpha = 2$,

$$F_2 = (A_1|A_2) = \left(\begin{array}{cc|cc} 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 \end{array}\right),$$

and

$$F = (F_1|F_2|F_3) = \left(\begin{array}{cccccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 4 & 3 & 6 & 5 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 2 & 4 & 3 & 6 & 5 \end{array}\right).$$

Combinatorial (n, m)-3-switch code

Example

Let $n = 6$, then $\alpha = 2$,

$$F_2 = (A_1|A_2) = \left(\begin{array}{cc|cc} 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 \end{array}\right),$$

and

$$F = (F_1|F_2|F_3) = \left(\begin{array}{cccccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 4 & 3 & 6 & 5 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 2 & 4 & 3 & 6 & 5 \end{array}\right).$$

The construction is generalized for every $\ell \geq 3$

Theorem

If $n = (\ell - 2)\alpha^2 + (\ell - 2)\alpha$ *then there exists an (explicit)* (n, m)*-*ℓ*-switch code with* $m = \ell n - (\ell - 1)(\ell - 2)\alpha \approx \ell n - (\ell - 1)\sqrt{(\ell - 2)n}$.

Conclusion

- We constructed the best known binary switch code for the case $n = k$.
- We presented consecutive switch codes, which can achieve better rates and yet recover a common type of request sets.
- We showed the tight lower bound $m = 2n - 1$ for combinatorial 2-consecutive switch codes.
- We constructed combinatorial ℓ-consecutive switch codes, with $m \approx \ell n - (\ell - 1)\sqrt{(\ell - 2)n}$.

Open Problems

- Find lower bounds on m for switch codes/combinatorial consecutive switch codes/consectutive switch codes.
- Construct combinatorial consecutive switch codes where ℓ is not fixed.
- Construct consecutive switch codes in the computational (not combinatorial) case.

Open Problems

- Find lower bounds on m for switch codes/combinatorial consecutive switch codes/consectutive switch codes.
- Construct combinatorial consecutive switch codes where ℓ is not fixed.
- Construct consecutive switch codes in the computational (not combinatorial) case.

Thank You!