

Semi-Constrained Systems

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- Motivation and Background

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- Current research

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- Rate penalty is reduced.

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Example

Let $\Sigma = \{0, 1\}$, $k = 2$, and $w = 011101$. Then, $|w| = 6$ and $\text{fr}_w^k = (0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5})$.

Definition of SCS

Definition (SCS)

Let $\mathcal{P}(\Sigma^k)$ denote the set of all probability measures on Σ^k . Let $\Gamma \subseteq \mathcal{P}(\Sigma^k)$ be a closed and convex set. A semiconstrained system (SCS), Γ , is the set

$$\mathcal{B}(\Gamma) = \left\{ w \in \Sigma^* : \text{fr}_w^k \in \Gamma \right\}.$$

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Definition (Capacity)

The capacity of $\mathcal{B}(\Gamma)$ defined as

$$\text{cap}(\mathcal{B}(\Gamma)) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 |\mathcal{B}_n(\Gamma)|.$$

Example $((0, k, p)$ -RLL SCS)

Let $\Sigma = \{0, 1\}$ and let $0 \leq p \leq 1$. We define the $(0, k, p)$ -RLL SCS as the set

$$\Gamma = \left\{ \mu \in \mathcal{P}(\Sigma^{k+1}) : \mu(1^{k+1}) \leq p \right\}.$$

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Let $\Sigma = \{0, 1\}$ and let

$$\Gamma = \left\{ \mu \in \mathcal{P}(\Sigma^2) : \mu(00) \leq \frac{1}{2}, \mu(11) \leq \frac{1}{2}, \mu(01) = 0 \right\}.$$

Example

Let $\Sigma = \{0, 1, 2, 3\}$ and let

$$\Gamma = \left\{ \mu \in \mathcal{P}(\Sigma^2) : \begin{aligned} &\mu(11) = 0, \\ &\mu(20) = \mu(30) = \mu(21) = \mu(31) = 0, \\ &\mu(00) + \mu(10) + \mu(01) \geq 0.25 \end{aligned} \right\}.$$

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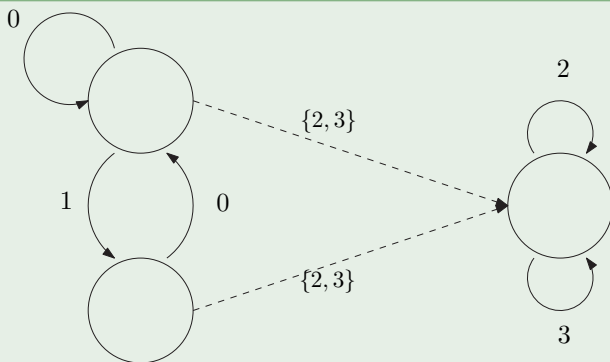


Figure: Graph for SCS Γ .

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- $\text{cap}(\mathcal{B}(\Gamma)) < 1$.
- A connected component with capacity 1.

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 - This work: general probabilistic and deterministic capacity achieving schemes.

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- $\text{cap}(\Gamma) = \log_2 |\Sigma| + \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 p_n$.
- For a SCS $\Gamma \subseteq \mathcal{P}(\Sigma^k)$ we obtain from large deviations theory

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 p_n \leq - \inf_{\nu \in \bar{\Gamma}} I(\nu),$$
$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log_2 p_n \geq - \inf_{\nu \in \Gamma^\circ} I(\nu).$$

where $I(\cdot)$ is given in terms of the K-L divergence.

Two Interesting Examples

Example (1)

Let $\mathcal{B}(\Gamma)$ be an SCS with $\Sigma = \{0, 1\}$ and $\Gamma = \{\mu \in \mathcal{P}(\Sigma) : \mu(0) = \mu(1) = \frac{1}{2}\}$. For an even number, n , $|\mathcal{B}_n(\Gamma)| = \binom{n}{n/2}$ but for an odd n , $|\mathcal{B}_n(\Gamma)| = 0$. Thus, $\text{cap}(\Gamma) > 0$.

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The limit does not exist.

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Example (2)

Let Γ be an SCS with $\Sigma = \{0, 1\}$ and $\Gamma = \{\mu \in \mathcal{P}(\Sigma) : \mu(0) = r, \mu(1) = 1 - r\}$ where $r \in [0, 1]$ is an irrational number. Since the capacity defined over finite sequences, for every n we obtain $\mathcal{B}_n(\Gamma) = \emptyset$, which implies $\text{cap}(\Gamma) = -\infty$.

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For every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log_2 |\mathcal{B}_n(\mathbb{B}_\epsilon(\Gamma))| > 0$$

exists. The second example shows that

$$\lim_{\epsilon \rightarrow 0} \text{cap}(\mathbb{B}_\epsilon(\Gamma)) \neq \text{cap}(\Gamma).$$

Weak SCS - Relaxing the constraints

Definition

A *weak semiconstrained system (WSCS)*, $\bar{\mathcal{B}}(\Gamma)$, is defined by

$$\bar{\mathcal{B}}(\Gamma) = \left\{ \omega \in \Sigma^* : \text{fr}_w^k \in \mathbb{B}_{\xi(|\omega|)}(\Gamma) \right\}$$

where $\xi : \mathbb{N} \rightarrow \mathbb{R}$, $\xi(n) = o(1)$ and $\xi(n) = \Omega\left(\frac{1}{n}\right)$.

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Theorem

For WSCS the limit exists and equals to $-\inf_{\nu \in \Gamma} I(\nu)$. If $\mathcal{B}(\Gamma) \neq \emptyset$ then the capacity is continuous with respect to the restrictions.

Bounds on the Capacity - A Case Study

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- The lower bound is calculating using large deviations theory.

Asymptotic Bounds

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Theorem

If $p \leq \frac{1}{2^{k+1}}$ then

$$C_{k,p} \leq 1 - \frac{1}{3 - 2^{-k+1}} \left(\frac{\log_2 e}{2^{k+1}} + p(k+1) - p \log_2 \frac{e}{p} \right).$$

$$C_{k,p} \geq 1 - \frac{1-p}{2^{k+1}-1} \log_2 \left(\frac{2-2p}{1+2p(2^k-1)} \right) - p \log_2 \left(\frac{2p(2^{k+1}-1)}{1+2p(2^k-1)} \right).$$

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- For $(0, k, p)$ -RLL SCS $p = p(k) \leq \frac{1}{2^{k+1}}$.
- Denote by $c = \lim_{k \rightarrow \infty} \frac{p}{2^{-(k+1)}}$.

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Theorem

For $p = p(k)$,

$$\frac{b_L}{2^{k+1}}(1 + o(1)) \leq 1 - C_{k,p} \leq \frac{b_U}{2^{k+1}}(1 + o(1))$$

where

$$b_L = \frac{3 - \sqrt{1 + 8c}}{4} \log_2 e - c \log_2 \left(\frac{1 + 4c + \sqrt{1 + 8c}}{8c} \right)$$

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- $1.35 \leq \frac{b_U}{b_L} \leq 1.52.$

Probabilistic Encoding Scheme

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- The scheme relies on LD theory.

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- Step 2: construct a Markov chain on the binary De-Bruijn graph of order $k - 1$ with stationary distribution on edges $p = (p_0, p_1, \dots, p_{2^k-1})$.
- Step 3: Partition and bias the Bernoulli $\frac{1}{2}$ input bits.

Encoder Recipe - Partitioning

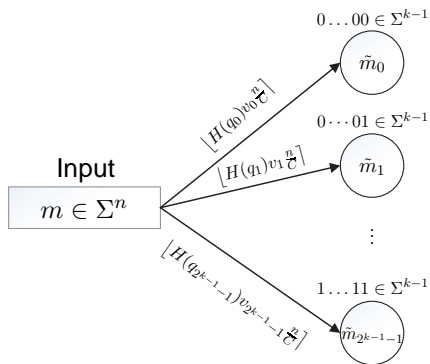


Figure: Partitioning.

Encoder Recipe - Biasing

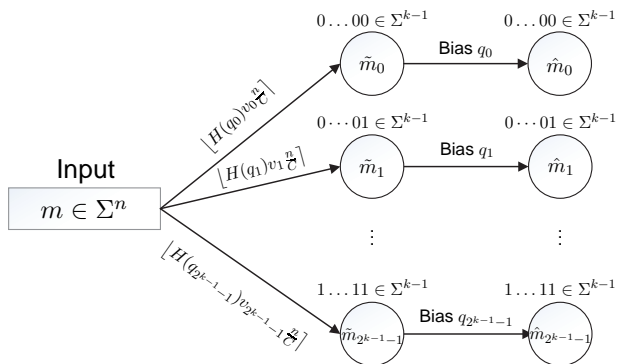


Figure: Biasing.

Encoder Recipe - Example for Graph Walking

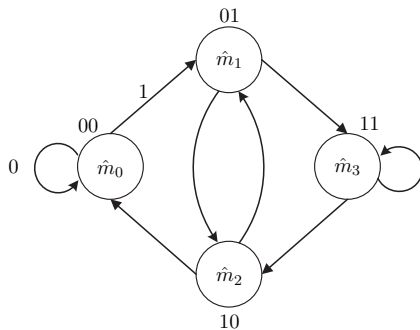


Figure: Example for Graph Walking, $k = 3$.

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- The encoder may fail but $\Pr_{fail} \rightarrow 0$ as $n \rightarrow \infty$.

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- Let $\Gamma \subseteq \mathcal{P}(\Sigma^k)$ be closed and convex. If Γ is **fat** then

$$\text{cap}(\Gamma) = \log |\Sigma| - \inf_{\eta \in \Gamma} I(\eta)$$

where $I(\cdot)$ is given in terms of the K-L divergence.

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Theorem (Lind and Marcus (95), C.3)

Let $\phi : \Sigma^{\mathbb{N}} \rightarrow X$ where $X = \phi(\Sigma^{\mathbb{N}})$ be an encoding function with finite memory and finite anticipation. Then X can be represented by a graph (sofic shift).

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- Convert a SCS into a fully constrained system.

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For any $\epsilon > 0$ define

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where $\|\cdot\|_{TV}$ is the total variation norm.

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- Γ -admissible $\epsilon \rightarrow \Gamma_\epsilon \neq \emptyset$ and fat.

Construction A

Let Γ be a fat SCS, for every $m \in \mathbb{N}$ we construct $\mathcal{R}_m(\Gamma) \subseteq \Sigma^*$ by defining

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Theorem

Let Γ be a fat SCS. Then for any Γ -admissible $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that for all $m > M_\epsilon$

$$\mathcal{R}_m(\Gamma_\epsilon) \subseteq \mathcal{B}(\Gamma).$$

Characteristics of Block Encoders

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Sliding Window Encoders

Construction B

Let Γ be a fat SCS. For every $m \in \mathbb{N}$ we construct $N_m(\Gamma) \subseteq \Sigma^*$ by defining

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Theorem

Let Γ be a fat SCS. Then for any Γ -admissible $\epsilon > 0$ and for every $m \geq k$ we have

$$N_m(\Gamma_\epsilon) \subseteq^e \mathcal{B}(\Gamma)$$

where \subseteq^e means $|N_m(\Gamma_\epsilon) \setminus \mathcal{B}(\Gamma)| < \infty$.

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Example for Cons. A and Cons. B

Example

Consider the $(0, 1, 0.205)$ -RLL SCS in which $\Sigma = \{0, 1\}$ and $\Gamma = \{\mu \in \mathcal{P}(\Sigma^2) : \mu(11) \leq 0.205\}$.

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	Cons. A	Cons. B
m	81	5
states	1	16
edges	Exponential	32

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sketch.

- Let $\alpha \in \Sigma^*$.
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- The graph is strongly connected.
- There exists an Eulerian cycle with α as a subword.



Corollary

Let Γ be a fat SCS. Then for every $\alpha \in \Sigma^$ there exists $\beta \in \Sigma^*$ such that $\alpha\beta \in \mathcal{B}(\Gamma)$.*

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Corollary

Let Γ be a fat SCS. Let Γ' be a fully constrained system such that $\mathcal{B}(\Gamma) \subseteq \mathcal{B}(\Gamma')$. Then $\mathcal{B}(\Gamma') = \Sigma^$.*

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Definition

Let $\Gamma \subseteq \mathcal{P}_{si}(\Sigma^k)$ be a SCS. The essential part of Γ is defined as

$$ess(\Gamma) := \{\eta \in \Gamma : \mathcal{B}(\eta) \neq \emptyset\}.$$

Fully Constrained Systems Containing a SCS

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Definition

For a SCS Γ , let $G_{ess}(\Gamma)$ be the graph with $V = \Sigma^{k-1}$ and each $e \in E$ corresponds to a k -tuple. $e \in E$ if $\exists \eta \in ess(\Gamma)$ with $\eta(e) > 0$.

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- In general $\text{cap}(\Gamma) \leq \text{cap}(\mathcal{L}(G_{ess}(\Gamma)))$.
- $\text{cap}(\Gamma) = 0$ iff $\text{cap}(\mathcal{L}(G_{ess}(\Gamma))) = 0$.

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End

Thank You!