Semi-Constrained Systems

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Outline

- Motivation and Background

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- Capacity of semiconstrained systems

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- Convergence rate of the capacity

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- Current research

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- Rate penalty is reduced.

Notation and Definitions

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Example

Let $\Sigma = \{0, 1\}$, $k = 2$, and $w = 011101$. Then, $|w| = 6$ and $\mathrm{fr}_w^k = (0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5})$.

Definition of SCS

Definition (SCS)

Let $\mathcal{P}(\Sigma^k)$ denote the set of all probability measures on Σ^k. Let $\Gamma \subseteq \mathcal{P}(\Sigma^k)$ be a closed and convex set. A semiconstrained system (SCS), Γ, is the set

$$\mathcal{B}(\Gamma) = \left\{ w \in \Sigma^* \ : \ \mathrm{fr}_w^k \in \Gamma \right\}.$$

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Definition (Capacity)

The capacity of $\mathcal{B}(\Gamma)$ defined as

$$\mathsf{cap}(\mathcal{B}(\Gamma)) := \limsup_{n \to \infty} \frac{1}{n} \log_2 |\mathcal{B}_n(\Gamma)|.$$

Example ($(0,k,p)$-RLL SCS)

Let $\Sigma = \{0,1\}$ and let $0 \leqslant p \leqslant 1$. We define the $(0,k,p)$-RLL SCS as the set

$$\Gamma = \left\{ \mu \in \mathcal{P}(\Sigma^{k+1}) \ : \ \mu(1^{k+1}) \leqslant p \right\}.$$

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Let $\Sigma = \{0,1\}$ and let

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Let $\Sigma = \{0, 1, 2, 3\}$ and let

$$\begin{aligned}\Gamma = \big\{\mu \in \mathcal{P}(\Sigma^2) \ : \ &\mu(11) = 0, \\ &\mu(20) = \mu(30) = \mu(21) = \mu(31) = 0, \\ &\mu(00) + \mu(10) + \mu(01) \geqslant 0.25\big\} .\end{aligned}$$

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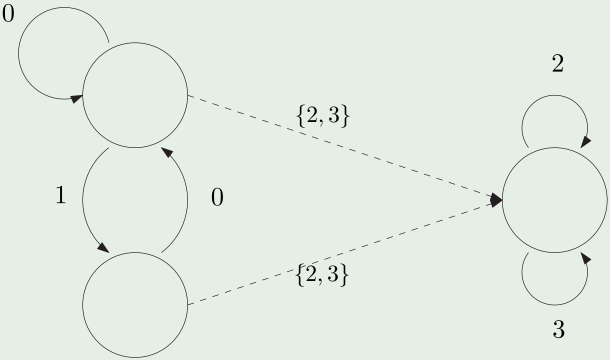

Figure: Graph for SCS Γ.

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- cap$(\mathcal{B}(\Gamma)) < 1$.

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- $\mathsf{cap}(\mathcal{B}(\Gamma)) < 1$.
- A connected component with capacity 1.

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  - This work: general probabilistic and deterministic capacity achieving schemes.

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- $\mathrm{cap}(\Gamma) = \log_2 |\Sigma| + \limsup_{n\to\infty} \frac{1}{n} \log_2 p_n$.
- For a SCS $\Gamma \subseteq \mathcal{P}(\Sigma^k)$ we obtain from large deviations theory

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 p_n \leqslant - \inf_{\nu\in\overline{\Gamma}} I(\nu),$$
$$\liminf_{n\to\infty} \frac{1}{n} \log_2 p_n \geqslant - \inf_{\nu\in\Gamma^\circ} I(\nu).$$

where $I(\cdot)$ is given in terms of the K-L divergence.

Two Interesting Examples

Example (1)

Let $\mathcal{B}(\Gamma)$ be an SCS with $\Sigma = \{0, 1\}$ and $\Gamma = \left\{\mu \in \mathcal{P}(\Sigma) \ : \ \mu(0) = \mu(1) = \frac{1}{2}\right\}$. For an even number, n, $|\mathcal{B}_n(\Gamma)| = \binom{n}{n/2}$ but for an odd n, $|\mathcal{B}_n(\Gamma)| = 0$. Thus, $\mathsf{cap}(\Gamma) > 0$.

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The limit does not exist.

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Example (2)

Let Γ be an SCS with $\Sigma = \{0, 1\}$ and $\Gamma = \{\mu \in \mathcal{P}(\Sigma) \ : \ \mu(0) = r, \mu(1) = 1 - r\}$ where $r \in [0, 1]$ is an irrational number. Since the capacity defined over finite sequences, for every n we obtain $\mathcal{B}_n(\Gamma) = \emptyset$, which implies $\mathsf{cap}(\Gamma) = -\infty$.

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For every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log_2 |\mathcal{B}_n (\mathbb{B}_\epsilon(\Gamma))| > 0$$

exists. The second example shows that

$$\lim_{\epsilon \to 0} \mathsf{cap}(\mathbb{B}_\epsilon(\Gamma)) \neq \mathsf{cap}(\Gamma).$$

Weak SCS - Relaxing the constraints

Definition

A *weak semiconstrained system (WSCS)*, $\overline{\mathcal{B}}(\Gamma)$, is defined by

$$\overline{\mathcal{B}}(\Gamma) = \left\{ \boldsymbol{\omega} \in \Sigma^* \ : \ \mathrm{fr}_w^k \in \mathbb{B}_{\xi(|\boldsymbol{\omega}|)}(\Gamma) \right\}$$

where $\xi : \mathbb{N} \to \mathbb{R}$, $\xi(n) = o(1)$ and $\xi(n) = \Omega(\frac{1}{n})$.

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Theorem

For WSCS the limit exists and equals to $-\inf_{\nu \in \Gamma} I(\nu)$*. If* $\mathcal{B}(\Gamma) \neq \emptyset$ *then the capacity is continuous with respect to the restrictions.*

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- The lower bound is calculating using large deviations theory.

Asymptotic Bounds

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Theorem

If $p \leqslant \frac{1}{2^{k+1}}$ *then*

$$C_{k,p} \leqslant 1 - \frac{1}{3-2^{-k+1}} \left(\frac{\log_2 e}{2^{k+1}} + p(k+1) - p\log_2 \frac{e}{p} \right).$$

$$C_{k,p} \geqslant 1 - \frac{1-p}{2^{k+1}-1} \log_2 \left(\frac{2-2p}{1+2p(2^k-1)} \right) - p\log_2 \left(\frac{2p(2^{k+1}-1)}{1+2p(2^k-1)} \right).$$

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- For $(0, k, p)$-RLL SCS $p = p(k) \leqslant \frac{1}{2^{k+1}}$.
- Denote by $c = \lim_{k\to\infty} \frac{p}{2^{-(k+1)}}$.

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Theorem

For $p = p(k)$*,*

$$\frac{b_{\mathrm{L}}}{2^{k+1}}(1+o(1)) \leqslant 1 - C_{k,p} \leqslant \frac{b_{\mathrm{U}}}{2^{k+1}}(1+o(1))$$

where

$$b_{\mathrm{L}} = \frac{3-\sqrt{1+8c}}{4}\log_2 e - c\log_2\left(\frac{1+4c+\sqrt{1+8c}}{8c}\right)$$

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- $1.35 \leqslant \frac{b_{\mathrm{U}}}{b_{\mathrm{L}}} \leqslant 1.52$.

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- Step 2: construct a Markov chain on the binary De-Bruijn graph of order $k-1$ with stationary distribution on edges $p = (p_0, p_1, \ldots, p_{2^k-1})$.
- Step 3: Partition and bias the Bernoulli $\frac{1}{2}$ input bits.

Encoder Recipe - Partitioning

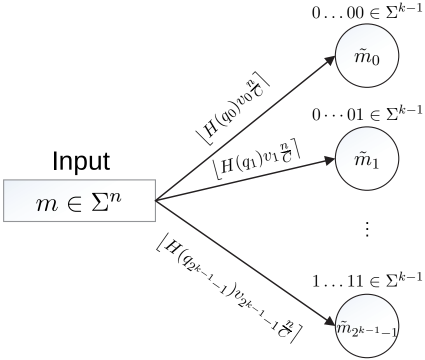
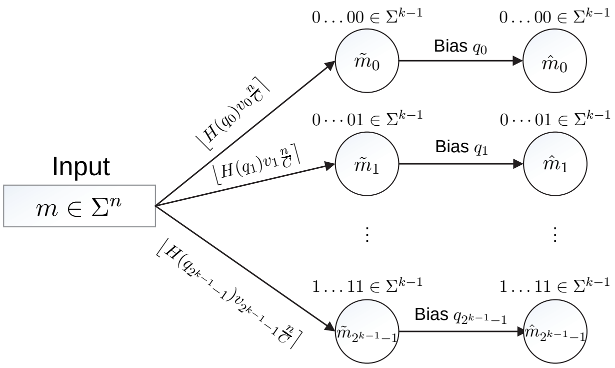


Figure: Partitioning.

Encoder Recipe - Biasing



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Encoder Recipe - Example for Graph Walking

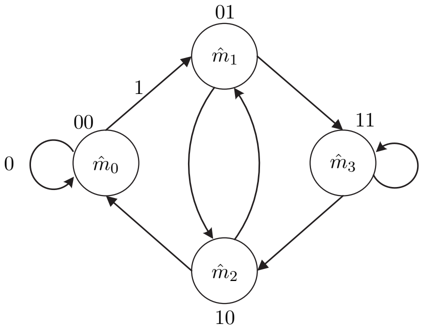

Figure: Example for Graph Walking, $k = 3$.

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- The encoder may fail but $\Pr_{fail} \to 0$ as $n \to \infty$.

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Theorem (Lind and Marcus (95), C.3)

Let $\phi : \Sigma^{\mathbb{N}} \rightarrow X$ where $X = \phi(\Sigma^{\mathbb{N}})$ be an encoding function with finite memory and finite anticipation. Then X can be represented by a graph (sofic shift).

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Definition

For any $\epsilon > 0$ define

$$\Gamma_\epsilon := \left\{ \eta \in \mathcal{P}(\Sigma^k) \ : \ \inf_{\mu \in \Gamma^c} ||\eta - \mu||_{TV} > \epsilon \right\}$$

where $|| \cdot ||_{TV}$ is the total variation norm.

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- Fat condition $\to \Gamma$ can be shrunk slightly.

Definition

For any $\epsilon > 0$ define

$$\Gamma_\epsilon := \left\{ \eta \in \mathcal{P}(\Sigma^k) \;:\; \inf_{\mu \in \Gamma^c} ||\eta - \mu||_{TV} > \epsilon \right\}$$

where $||\cdot||_{TV}$ is the total variation norm.

- Γ-admissible $\epsilon \to \Gamma_\epsilon \neq \emptyset$ and fat.

Construction A

Let Γ be a fat SCS, for every $m \in \mathbb{N}$ we construct $\mathcal{R}_m(\Gamma) \subseteq \Sigma^*$ by defining

$$\mathcal{R}_m(\Gamma) := (\mathcal{B}_m(\Gamma))^*.$$

Block Encoders for SCS

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Theorem

Let Γ be a fat SCS. Then for any Γ-admissible $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that for all $m > M_\epsilon$

$$\mathcal{R}_m(\Gamma_\epsilon) \subseteq \mathcal{B}(\Gamma).$$

Characteristics of Block Encoders

- $\lim_{m\to\infty} \mathsf{cap}(\mathcal{R}_m(\Gamma_\epsilon)) = \mathsf{cap}(\mathcal{B}(\Gamma_\epsilon))$.

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- $M_\epsilon = \Omega(\frac{1}{\epsilon})$.

Sliding Window Encoders

Construction B

Let Γ be a fat SCS. For every $m \in \mathbb{N}$ we construct $N_m(\Gamma) \subseteq \Sigma^*$ by defining

$$N_m(\Gamma) := \{w \in \Sigma^* \ : \ \mathrm{sub}_m(w) \subseteq \mathcal{B}(\Gamma)\} .$$

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Theorem

Let Γ be a fat SCS. Then for any Γ-admissible $\epsilon > 0$ and for every $m \geqslant k$ we have

$$N_m(\Gamma_\epsilon) \subseteq^e \mathcal{B}(\Gamma)$$

where $\subseteq^e$ means $|N_m(\Gamma_\epsilon) \setminus \mathcal{B}(\Gamma)| < \infty$.

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Example for Cons. A and Cons. B

Example

Consider the $(0, 1, 0.205)$-RLL SCS in which $\Sigma = \{0, 1\}$ and $\Gamma = \left\{\mu \in \mathcal{P}(\Sigma^2) \ : \ \mu(11) \leqslant 0.205\right\}$.

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	Cons. A	Cons. B
m	81	5
states	1	16
edges	Exponential	32

The Dual Question

- Can we approach a SCS's capacity from above?

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Theorem

Let $\mu \in \mathcal{P}(\Sigma^k)$ *be a positive, shift invariant, and rational measure. Then for every* $\alpha \in \Sigma^*$ *there exists* $\beta \in \Sigma^*$ *such that* $\mathrm{fr}^k_{\alpha\beta} = \mu$ *and the first* $(k-1)$ *tuple equals the last.*

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- There exists an Eulerian cycle with α as a subword.

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Corollary

Let Γ be a fat SCS. Let Γ' be a fully constrained system such that $\mathcal{B}(\Gamma) \subseteq \mathcal{B}(\Gamma')$. Then $\mathcal{B}(\Gamma') = \Sigma^*$.

Combining SCS with Combinatorial Constraints

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- Construction B works.

Fully Constrained Systems Containing a SCS

Definition

Let $\Gamma \subseteq \mathcal{P}_{si}(\Sigma^k)$ be a SCS. The essential part of Γ is defined as

$$ess(\Gamma) := \{\eta \in \Gamma \ : \ \mathcal{B}(\eta) \neq \emptyset\} \, .$$

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Definition

For a SCS Γ, let $G_{ess}(\Gamma)$ be the graph with $V = \Sigma^{k-1}$ and each $e \in E$ corresponds to a k-tuple. $e \in E$ if $\exists \eta \in \ ess(\Gamma)$ with $\eta(e) > 0$.

Essential Graph

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Let $\Gamma \subseteq \mathcal{P}_{si}(\Sigma^k)$ *be a SCS, then*

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- In general $\mathsf{cap}(\Gamma) \leqslant \mathsf{cap}(\mathcal{L}(G_{ess}(\Gamma)))$.
- $\mathsf{cap}(\Gamma) = 0$ iff $\mathsf{cap}(\mathcal{L}(G_{ess}(\Gamma))) = 0$.

Current Research

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End

Thank You!