Semi-Constrained Systems

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Motivation and Background
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- Capacity of semiconstrained systems
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- Current research
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- Semiconstrained systems (SCS) are a middle road.
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- Two common approaches.
- Semiconstrained systems (SCS) are a middle road.
- Rate penalty is reduced.
Notation and Definitions

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Example

Let $\Sigma = \{0, 1\}$, $k = 2$, and $w = 011101$. Then, $|w| = 6$ and $fr^k_w = (0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5})$. 
Definition (SCS)

Let $\mathcal{P}(\Sigma^k)$ denote the set of all probability measures on $\Sigma^k$. Let $\Gamma \subseteq \mathcal{P}(\Sigma^k)$ be a closed and convex set. A semiconstrained system (SCS), $\Gamma$, is the set

$$\mathcal{B}(\Gamma) = \left\{ w \in \Sigma^* : \text{fr}_w^k \in \Gamma \right\}.$$
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**Definition (Capacity)**

The capacity of $\mathcal{B}(\Gamma)$ defined as

$$
cap(\mathcal{B}(\Gamma)) := \limsup_{n \to \infty} \frac{1}{n} \log_2 |\mathcal{B}_n(\Gamma)|.
$$
Example \(((0, k, p)\text{-RLL SCS})\)

Let \( \Sigma = \{0, 1\} \) and let \( 0 \leq p \leq 1 \). We define the \((0, k, p)\text{-RLL SCS}\) as the set

\[
\Gamma = \left\{ \mu \in \mathcal{P}(\Sigma^{k+1}) : \mu(1^{k+1}) \leq p \right\}.
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Examples

**Example** $((0, k, p)\text{-RLL SCS})$

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**Example**

Let $\Sigma = \{0, 1\}$ and let

$$\Gamma = \left\{ \mu \in \mathcal{P}(\Sigma^2) : \mu(00) \leq \frac{1}{2}, \mu(11) \leq \frac{1}{2}, \mu(01) = 0 \right\}.$$
Let $\Sigma = \{0, 1, 2, 3\}$ and let

$$\Gamma = \{ \mu \in \mathcal{P}(\Sigma^2) : \mu(11) = 0,$$
$$\mu(20) = \mu(30) = \mu(21) = \mu(31) = 0,$$
$$\mu(00) + \mu(10) + \mu(01) \geq 0.25 \}.$$
Examples

Figure: Graph for SCS $\Gamma$. 
Example

Let $\Sigma = \{0, 1, 2, 3\}$ and let

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- $\text{cap}(\mathcal{B}(\Gamma)) < 1$.
- A connected component with capacity 1.
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- **Encoding scheme.**
  - This work: general probabilistic and deterministic capacity achieving schemes.
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- Let $p_n$ denote the probability that a random string of length $n$ is in $B_n(\Gamma)$.
- We have $|B_n(\Gamma)| = p_n|\Sigma|^n$.
- $\text{cap}(\Gamma) = \log_2|\Sigma| + \lim \sup_{n \to \infty} \frac{1}{n} \log_2 p_n$. 
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$\text{cap}(\Gamma) = \log_2 |\Sigma| + \limsup_{n \to \infty} \frac{1}{n} \log_2 p_n$.

For a SCS $\Gamma \subseteq \mathcal{P}(\Sigma^k)$ we obtain from large deviations theory

$$\limsup_{n \to \infty} \frac{1}{n} \log_2 p_n \leq - \inf_{\nu \in \Gamma} I(\nu),$$

$$\liminf_{n \to \infty} \frac{1}{n} \log_2 p_n \geq - \inf_{\nu \in \Gamma^c} I(\nu).$$

where $I(\cdot)$ is given in terms of the K-L divergence.
Two Interesting Examples

Example (1)

Let $B(\Gamma)$ be an SCS with $\Sigma = \{0, 1\}$ and
$\Gamma = \{\mu \in \mathcal{P}(\Sigma) : \mu(0) = \mu(1) = \frac{1}{2}\}$. For an even number, $n$, $|B_n(\Gamma)| = \binom{n}{n/2}$ but for an odd $n$, $|B_n(\Gamma)| = 0$. Thus, $\text{cap}(\Gamma) > 0$. 
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Two Interesting Examples

Example (2)

Let $\Gamma$ be an SCS with $\Sigma = \{0, 1\}$ and

$\Gamma = \{\mu \in \mathcal{P}(\Sigma) : \mu(0) = r, \mu(1) = 1 - r\}$

where $r \in [0, 1]$ is an irrational number. Since the capacity defined over finite sequences, for every $n$ we obtain $B_n(\Gamma) = \emptyset$, which implies $\text{cap}(\Gamma) = -\infty$. 

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For every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log_2 |B_n(B_\epsilon(\Gamma))| > 0$$

exists. The second example shows that

$$\lim_{\epsilon \to 0} \text{cap}(B_\epsilon(\Gamma)) \neq \text{cap}(\Gamma).$$
A weak semiconstrained system (WSCS), $\overline{B}(\Gamma)$, is defined by

$$\overline{B}(\Gamma) = \left\{ \omega \in \Sigma^* : \text{fr}^k_w \in B_{\xi(|\omega|)}(\Gamma) \right\}$$

where $\xi : \mathbb{N} \rightarrow \mathbb{R}$, $\xi(n) = o(1)$ and $\xi(n) = \Omega\left(\frac{1}{n}\right)$. 
Definition

A weak semiconstrained system (WSCS), $\overline{B}(\Gamma)$, is defined by

$$\overline{B}(\Gamma) = \left\{ \omega \in \Sigma^* : \frac{fr^k_w}{\xi(|\omega|)}(\Gamma) \right\}$$

where $\xi : \mathbb{N} \rightarrow \mathbb{R}$, $\xi(n) = o(1)$ and $\xi(n) = \Omega(\frac{1}{n})$.

Theorem

For WSCS the limit exists and equals to $- \inf_{\nu \in \Gamma} I(\nu)$. If $B(\Gamma) \neq \emptyset$ then the capacity is continuous with respect to the restrictions.
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We find bounds on the capacity.
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The capacity expression is given by an optimization problem.

We find bounds on the capacity.

For the upper bound we employ a bound by Janson.

The lower bound is calculated using large deviations theory.
Asymptotic Bounds

- We denote the capacity of $(0, k, p)$-RLL SCS as $C_{k,p}$. 
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- If \(p \geq \frac{1}{2^{k+1}}\) then \(C_{k,p} = 1\).
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**Theorem**

If \(p \leq \frac{1}{2^{k+1}}\) then

\[ C_{k,p} \leq 1 - \frac{1}{3 - 2^{-k+1}} \left( \frac{\log_2 e}{2^{k+1}} + p(k + 1) - p \log_2 \frac{e}{p} \right) . \]

\[ C_{k,p} \geq 1 - \frac{1 - p}{2^{k+1} - 1} \log_2 \left( \frac{2 - 2p}{1 + 2p(2^k - 1)} \right) - p \log_2 \left( \frac{2p(2^{k+1} - 1)}{1 + 2p(2^k - 1)} \right) . \]
Asymptotic Bounds

- As $k \to \infty$ the capacity goes to 1.
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- For fully constrained systems it is known that

$$1 - C_{k,0} = \frac{\log_2 e}{4 \cdot 2^k} (1 + o(1)).$$
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  \[
  1 - C_{k,0} = \frac{\log_2 e}{4} \cdot \frac{1}{2^k} (1 + o(1)).
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- For $(0, k, p)$-RLL SCS $p = p(k) \leq \frac{1}{2^{k+1}}$. 
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  \]
- For $(0, k, p)$-RLL SCS $p = p(k) \leq \frac{1}{2^{k+1}}$.
- Denote by $c = \lim_{k \to \infty} \frac{p}{2^{-(k+1)}}$. 
Asymptotic Bounds

**Theorem**

For $p = p(k)$,

$$
\frac{b_L}{2^{k+1}}(1 + o(1)) \leq 1 - C_{k,p} \leq \frac{b_U}{2^{k+1}}(1 + o(1))
$$

where

$$
b_L = 3 - \sqrt{1 + 8c} \frac{\log_2 e - c \log_2 \left( \frac{1 + 4c + \sqrt{1 + 8c}}{8c} \right)}{4}
$$

and

$$
b_U = (1 + c)(1 - H(\frac{1}{c + 1}))
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- $1.35 \leq \frac{b_U}{b_L} \leq 1.52$. 
Probabilistic Encoding Scheme

Inspired by a coding scheme briefly sketched in a paper by S. Aviran, P. Siegel, and J. Wolf (2005).
Probabilistic Encoding Scheme

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- The scheme relies on LD theory.
Assumptions: \( \Sigma = \{0, 1\} \) and the constraints are on \( k \)-tuples.
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- \( p = (p_0, p_1, \ldots, p_{2^k-1}) \) the \( k \)-tuples empirical distribution.
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- $C$ the capacity of the WSCS.
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Step 2: construct a Markov chain on the binary De-Bruijn graph of order \( k - 1 \) with stationary distribution on edges \( p = (p_0, p_1, \ldots, p_{2^k - 1}) \).
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Step 3: Partition and bias the Bernoulli \( \frac{1}{2} \) input bits.
Encoder Recipe - Partitioning

Input
\[ m \in \Sigma^n \]

0\ldots00 \in \Sigma^{k-1}

0\ldots01 \in \Sigma^{k-1}

\[ \tilde{m}_0 \]

\[ \tilde{m}_1 \]

\[ \vdots \]

1\ldots11 \in \Sigma^{k-1}

\[ \tilde{m}_{2^{k-1}-1} \]

**Figure:** Partitioning.

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Encoder Recipe - Biasing

\[ m \in \Sigma^n \]

\( \hat{m}_0 \) \hspace{1cm} 0...00 \in \Sigma^{k-1} \hspace{1cm} 0...00 \in \Sigma^{k-1} \)

\( \hat{m}_1 \) \hspace{1cm} 0...01 \in \Sigma^{k-1} \hspace{1cm} 0...01 \in \Sigma^{k-1} \)

\( \hat{m}_{2k-1-1} \) \hspace{1cm} 1...11 \in \Sigma^{k-1} \hspace{1cm} 1...11 \in \Sigma^{k-1} \)

\( [H(q_0)v_0 \mathbb{c}] \)

\( [H(q_1)v_1 \mathbb{c}] \)

\( [H(q_{2k-1-1})v_{2k-1-1} \mathbb{c}] \)

Bias \( q_0 \)

Bias \( q_1 \)

Bias \( q_{2k-1-1} \)

Figure: Biasing.
Figure: Example for Graph Walking, $k = 3$. 
Encoding scheme: Some comments

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- The coding scheme asymptotically achieves capacity.
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- The coding scheme asymptotically achieves capacity.
- The encoder may fail but \( \Pr_{\text{fail}} \rightarrow 0 \) as \( n \rightarrow \infty \).
Deterministic Encoders

- General, deterministic encoders.
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- $\Gamma$ is fat $\rightarrow$ $\Gamma$ can be shrunk.
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- $\Gamma$ is fat $\rightarrow$ $\Gamma$ can be shrunk.
- Let $\Gamma \subseteq \mathcal{P}(\Sigma^k)$ be closed and convex. If $\Gamma$ is fat then

$$\operatorname{cap}(\Gamma) = \log |\Sigma| - \inf_{\eta \in \Gamma} I(\eta)$$

where $I(\cdot)$ is given in terms of the K-L divergence.
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Theorem (Lind and Marcus (95), C.3)

Let $\phi : \Sigma^N \rightarrow X$ where $X = \phi(\Sigma^N)$ be an encoding function with finite memory and finite anticipation. Then $X$ can be represented by a graph (sofic shift).
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Let $\phi : \Sigma^N \rightarrow X$ where $X = \phi(\Sigma^N)$ be an encoding function with finite memory and finite anticipation. Then $X$ can be represented by a graph (sofic shift).

- Convert a SCS into a fully constrained system.
Sequence of fully-constrained systems.
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Fat condition $\Gamma$ can be shrunk slightly.
Sequence of fully-constrained systems.

Fat condition $\rightarrow \Gamma$ can be shrunk slightly.

**Definition**

For any $\epsilon > 0$ define

$$\Gamma_\epsilon := \left\{ \eta \in \mathcal{P}(\Sigma^k) : \inf_{\mu \in \Gamma^c} ||\eta - \mu||_{TV} > \epsilon \right\}$$

where $|| \cdot ||_{TV}$ is the total variation norm.
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**Definition**

For any $\epsilon > 0$ define

$$\Gamma_\epsilon := \left\{ \eta \in \mathcal{P}(\Sigma^k) : \inf_{\mu \in \Gamma^c} \|\eta - \mu\|_{TV} > \epsilon \right\}$$

where $\| \cdot \|_{TV}$ is the total variation norm.

$\Gamma$-admissible $\epsilon \rightarrow \Gamma_\epsilon \neq \emptyset$ and fat.
Construction A

Let $\Gamma$ be a fat SCS, for every $m \in \mathbb{N}$ we construct $\mathcal{R}_m(\Gamma) \subseteq \Sigma^*$ by defining

$$\mathcal{R}_m(\Gamma) := (\mathcal{B}_m(\Gamma))^*.$$
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Theorem

Let $\Gamma$ be a fat SCS. Then for any $\Gamma$-admissible $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that for all $m > M_\epsilon$

$$\mathcal{R}_m(\Gamma_\epsilon) \subseteq \mathcal{B}(\Gamma).$$
 Characteristics of Block Encoders

\[ \lim_{m \to \infty} \text{cap}(\mathcal{R}_m(\Gamma_\epsilon)) = \text{cap}(\mathcal{B}(\Gamma_\epsilon)). \]
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- \( \exists \text{ block-encoders with rate } \to \text{cap}(\mathcal{B}(\Gamma)) \).
- \( M_\epsilon = \Omega(\frac{1}{\epsilon}) \).
Sliding Window Encoders
Let $\Gamma$ be a fat SCS. For every $m \in \mathbb{N}$ we construct $N_m(\Gamma) \subseteq \Sigma^*$ by defining

$$N_m(\Gamma) := \{ w \in \Sigma^* : \text{sub}_m(w) \subseteq \mathcal{B}(\Gamma) \}.$$
Construction B

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Theorem

Let $\Gamma$ be a fat SCS. Then for any $\Gamma$-admissible $\epsilon > 0$ and for every $m \geq k$ we have

$$N_m(\Gamma_{\epsilon}) \subseteq^e \mathcal{B}(\Gamma)$$

where $\subseteq^e$ means $|N_m(\Gamma_{\epsilon}) \setminus \mathcal{B}(\Gamma)| < \infty$. 
Characteristics of Sliding Window Encoders

\[ \limsup_{m \to \infty} \text{cap}(N_m(\Gamma)) = \text{cap}(B(\Gamma)). \]
Characteristics of Sliding Window Encoders

- \( \limsup_{m \to \infty} \text{cap}(N_m(\Gamma)) = \text{cap}(B(\Gamma)) \).
- \( N_m(\Gamma_\epsilon) \subseteq^e B(\Gamma) \) depends only on the length of the word and not on \( m \).
Example

Consider the $(0, 1, 0.205)$-RLL SCS in which $\Sigma = \{0, 1\}$ and $\Gamma = \{\mu \in \mathcal{P}(\Sigma^2) : \mu(11) \leq 0.205\}.$
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$\text{cap}(\Gamma) \approx 0.98$. 

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SCS
Example for Cons. A and Cons. B

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<table>
<thead>
<tr>
<th></th>
<th>Cons. A</th>
<th>Cons. B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m)</td>
<td>81</td>
<td>5</td>
</tr>
<tr>
<td>states</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>edges</td>
<td>Exponential</td>
<td>32</td>
</tr>
</tbody>
</table>
The Dual Question

- Can we approach a SCS’s capacity from above?
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**Theorem**

Let $\mu \in \mathcal{P}(\Sigma^k)$ be a positive, shift invariant, and rational measure. Then for every $\alpha \in \Sigma^*$ there exists $\beta \in \Sigma^*$ such that $fr_{\alpha\beta}^k = \mu$ and the first $(k - 1)$ tuple equals the last.
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**Sketch.**

- Let \( \alpha \in \Sigma^* \).
- We construct a De-Bruijn graph with multiple parallel edges, according to \( \mu \).
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Let $\mu \in \mathcal{P}(\Sigma^k)$ be a positive, shift invariant, and rational measure. Then for every $\alpha \in \Sigma^*$ there exists $\beta \in \Sigma^*$ such that $fr^k_{\alpha\beta} = \mu$ and the first $(k - 1)$ tuple equals the last.

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- Let $\alpha \in \Sigma^*$.
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- The graph is strongly connected.
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**sketch.**

- Let $\alpha \in \Sigma^*$.
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- The graph is strongly connected.
- There exists an Eulerian cycle with $\alpha$ as a subword.
The Dual Question

**Corollary**

Let $\Gamma$ be a fat SCS. Then for every $\alpha \in \Sigma^*$ there exists $\beta \in \Sigma^*$ such that $\alpha \beta \in \mathcal{B}(\Gamma)$. 
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Corollary

Let $\Gamma$ be a fat SCS. Let $\Gamma'$ be a fully constrained system such that $\mathcal{B}(\Gamma) \subseteq \mathcal{B}(\Gamma')$. Then $\mathcal{B}(\Gamma') = \Sigma^*$. 
The fat condition means there are no 0 restrictions.
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A SCS $\Gamma$ is relatively fat (RF) if it is fat with respect to the allowed $k$-tuples probabilities.
Combining SCS with Combinatorial Constraints

- The fat condition means there are no 0 restrictions.
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The fat condition means there are no 0 restrictions. A SCS $\Gamma$ is relatively fat (RF) if it is fat with respect to the allowed $k$-tuples probabilities. Construction A does not work. Construction B works.
Let $\Gamma \subseteq \mathcal{P}_{si}(\Sigma^k)$ be a SCS. The essential part of $\Gamma$ is defined as

$$ess(\Gamma) := \{ \eta \in \Gamma : \mathcal{B}(\eta) \neq \emptyset \}.$$
Definition

Let $\Gamma \subseteq \mathcal{P}_{si}(\Sigma^k)$ be a SCS. The essential part of $\Gamma$ is defined as

$$ess(\Gamma) := \{ \eta \in \Gamma : B(\eta) \neq \emptyset \}.$$ 

Definition

For a SCS $\Gamma$, let $G_{ess}(\Gamma)$ be the graph with $V = \Sigma^{k-1}$ and each $e \in E$ corresponds to a $k$-tuple. $e \in E$ if $\exists \eta \in ess(\Gamma)$ with $\eta(e) > 0$. 
Let $\Gamma \subseteq P_{si}(\Sigma^k)$ be a SCS, then

$$\mathcal{B}(\Gamma) \subseteq \mathcal{L}(G_{ess}(\Gamma)).$$
Theorem

Let $\Gamma \subseteq P_{si}(\Sigma^k)$ be a SCS, then

$$B(\Gamma) \subseteq L(G_{ess}(\Gamma)).$$

Theorem

Let $\Gamma \subseteq P_{si}(\Sigma^k)$ be a convex SCS. Then $L(G_{ess}(\Gamma))$ is the unique smallest fully constrained system containing $B(\Gamma)$. 
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Let \( \Gamma \subseteq P_{si}(\Sigma^k) \) be a SCS, then

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Theorem

Let \( \Gamma \subseteq P_{si}(\Sigma^k) \) be a convex SCS. Then \( \mathcal{L}(G_{ess}(\Gamma)) \) is the unique smallest fully constrained system containing \( \mathcal{B}(\Gamma) \).

- In general \( \text{cap}(\Gamma) \leq \text{cap}(\mathcal{L}(G_{ess}(\Gamma))). \)
Theorem

Let $\Gamma \subseteq P_{si}(\Sigma^k)$ be a SCS, then

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Theorem

Let $\Gamma \subseteq P_{si}(\Sigma^k)$ be a convex SCS. Then $\mathcal{L}(G_{ess}(\Gamma))$ is the unique smallest fully constrained system containing $\mathcal{B}(\Gamma)$.

- In general $\text{cap}(\Gamma) \leq \text{cap}(\mathcal{L}(G_{ess}(\Gamma)))$.
- $\text{cap}(\Gamma) = 0$ iff $\text{cap}(\mathcal{L}(G_{ess}(\Gamma))) = 0$. 

Ohad Elishco

SCS
Current Research

- $d$-dimensional SCS
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Thank You!