On the VC-Dimension of Binary Error-Correcting Codes

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Joint work with Nir Weinberger and Ofer Shayevitz

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  Statistical Learning, Computational Geometry, ...

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- Shattered coordinate set

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{matrix}$$

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$$
\begin{array}{|ccc|cccc}
\color{blue}0 & \color{blue}0 & \color{blue}0 & 0 & 0 & 0 & 0 \\
\color{blue}1 & \color{blue}0 & \color{blue}0 & 1 & 0 & 1 & 1 \\
\color{blue}0 & \color{blue}1 & \color{blue}0 & 1 & 1 & 1 & 0 \\
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- **VC-dimension**: maximum size of a shattered coordinate set
- Fact: The VC-dimension of the above code is 3.

Two (Extremal) Examples

- $\mathcal{C}$ = Linear code of dimension k and with generator matrix

$$\mathbf{G} := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

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Let $\mathcal{C}$ be a length-n binary code with VC-dimension D. Then

$$\leq |\mathcal{C}| \leq$$

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Large Minimum Distance & Small VC-Dimension

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WANTED

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Examples

$\mathcal{C}$	Mini. Dist.	VCD
Linear codes	Large	Large
Hamming balls	Small	Small
Random codes	Large	Large

Asymptotic Problem Formulation

- Rate $R \in [0, 1]$ is (δ, d)-achievable if $\exists \{\mathcal{C}_n\}_{n=1}^{\infty}$ such that

$$
\begin{aligned}
\mathrm{dist}(\mathcal{C}_n) &\geq \delta n \\
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- In this talk: two upper bounds + two lower bounds

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Second MRRW bound (ignore VCD constraint)

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Corollary of Haussler's packing lemma (1995)

$$C(d,\delta) \le \frac{2d}{\delta+2d} \cdot h\left(\left\langle \frac{\delta+2d}{2} \right\rangle\right)$$

Here $\langle a \rangle \stackrel{\text{def}}{=} \min\{a, \frac{1}{2}\}$.

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PF. The Probabilistic Method – Linearity of Expectation.

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$$
\begin{array}{cccc|cc}
x_1 & x_2 & \cdots & x_{sn} & \cdots & x_n \\
y_1 & y_2 & \cdots & y_{sn} & \cdots & y_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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By Sauer–Shelah lemma

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- Fix $\mathbf{u} \in \mathcal{C}|_{[sn]}$, consider its possble suffixes $\mathcal{Z}(\mathbf{u})$

$$u_1 u_2 \cdots u_{sn} \begin{cases} \boxed{\begin{array}{ccc} u_{sn+1} & \cdots & u_n \\ y_{sn+1} & \cdots & y_n \\ \vdots & \vdots & \vdots \\ z_{sn+1} & \cdots & z_n \end{array}} \end{cases}$$

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$|\mathcal{Z}(\mathbf{u})|$: second MRRW bound.

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Upper bound via projection (H., Weinberger, Shayevitz'17)

$$C(d,\delta) \le \min_{0 \le s \le 1-2\delta} \left\{ s \cdot h\left(\left\langle \frac{d}{s} \right\rangle\right) + (1-s) \cdot R_{LP}\left(\frac{\delta}{1-s}\right) \right\}$$

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For $\mathcal{C} \subseteq \{0,1\}^n$, there exists a Hamming ball of radius dn with at least

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codewords in it.

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3. We can do better via a direct approach!

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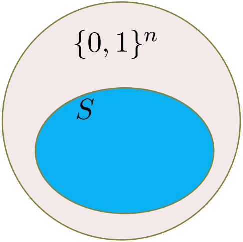
1. Pick $S \subseteq \{0,1\}^n$ with "nice" structure, e.g. constant weight, Markov type



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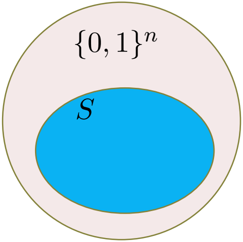

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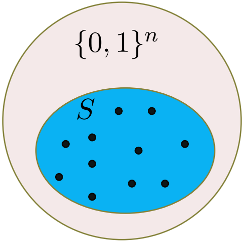

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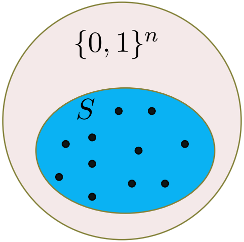
4. Combine above bounds and evaluate $n \to \infty$ asymptotics.



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Claim

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Proof: Suppose $\text{VCD}(\mathcal{C}) = dn$. Then $\exists\, \mathbf{x}, \mathbf{y} \in \mathcal{C}$ such that

$$\begin{array}{rcccccccc} \mathbf{x} = & (1 & 1 & 1 & \cdots & 1 & x_{dn+1} & \cdots & x_n) \\ \mathbf{y} = & (0 & 1 & 1 & \cdots & 1 & y_{dn+1} & \cdots & y_n) \end{array}$$

$$\delta n \leq \text{dist}(\mathbf{x}, \mathbf{y}) = 2wn - 2|\text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y})| \leq 2wn - 2(dn - 1)$$

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GV bound for constant weight codes

$$A(n, \delta n, wn) \geq \frac{\binom{n}{wn}}{\sum_{i=0}^{\delta n/2-1} \binom{wn}{i}\binom{n-wn}{i}}$$

Constant-weight lower bound (H., Weinberger, Shayevitz'17)

Let $d, \delta \in [0, \frac{1}{2}]$, and let $w = d + \frac{\delta}{2}$. Then

$$C(d, \delta) \geq \begin{cases} h(w) - \max\limits_{0 \leq x \leq \delta/2} \left[w\, h\left(\frac{x}{w}\right) + (1-w) h\left(\frac{x}{1-w}\right)\right], & w < \frac{1}{2} \\ 1 - h(\delta), & \text{otherwise.} \end{cases}$$

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Generalized GV [Kolesnik, Krachkovsky'91]

There exists an $(S, M, \delta n)$-code such that

$$M \ge \frac{|S|}{4\,|B|\,/|S|}$$

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Remark. Improved by Gu and Fuja (1993), Tolhuizen (1997, implied by Turán's theorem).

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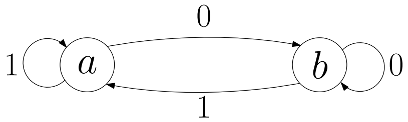

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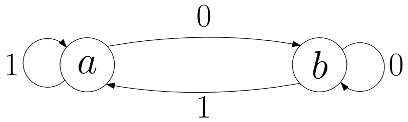

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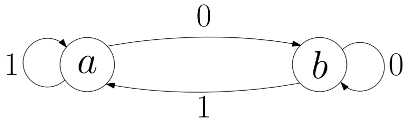

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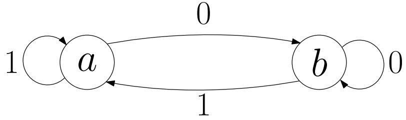

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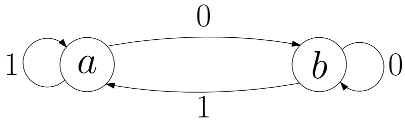

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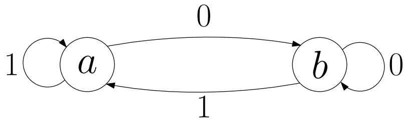

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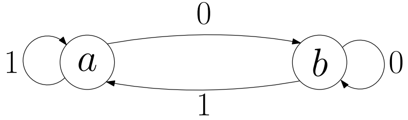

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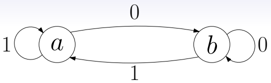

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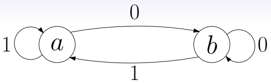

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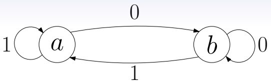

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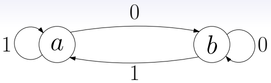

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$$\mathbf{E}_{P_\gamma}(f) = \sum_{e \in E_G} P_\gamma(e) f(e)$$

as the empirical average of f on the cycle γ.

Fact

For a cycle $\gamma = e_1 e_2 \cdots e_n \in \Gamma_n(G)$, the value $nE_{P_\gamma}(f) - f(e_1)$ is equal to the number of switches of the corresponding binary sequence $L_G(e_1)L_G(e_2)\cdots L_G(e_n)$.

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- $|S| = |\Gamma_n(G; f, [0,d))|$

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Lemma (Marcus, Roth'92)

Let G be a primitive graph and $f : E_G \to \mathbb{R}^k$ be a function on the edges of G. Let U be an open and nonempty subset of $\mathbb{R}^k$. Then

$$\lim_{n\to\infty} \frac{1}{n} \log |\Gamma_n(G; f, U)| = \sup_{P \in \mathcal{M}(G;f,U)} H_P(Y|X).$$

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This lemma is a consequence of well-known results on second-order types of Markov chains, cf. Boza’71, Davisson, Longo, Sgarro’81, Natarajan’85, Csiszár, Cover, Choi’87, and Csiszár’98.

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Lemma (Convex duality principle, Marcus, Roth'92)

Let G be a graph and let $f : E_G \to \mathbb{R}^k$, $g : E_G \to \mathbb{R}^l$ be functions on the edges of G. Set $\phi = [f, g] : E_G \to \mathbb{R}^{k+l}$. Then for any $\boldsymbol{r} \in \mathbb{R}^k$ and $\boldsymbol{s} \in \mathbb{R}^l$,

$$\sup_{\substack{P \in \mathcal{M}(G): \\ \mathbf{E}_P(f) = \boldsymbol{r} \\ \mathbf{E}_P(g) \le \boldsymbol{s}}} H_P(Y|X) = \inf_{\substack{\boldsymbol{x} \in \mathbb{R}^k \\ \boldsymbol{z} \in \mathbb{R}^l_{\ge 0}}} \{\boldsymbol{x} \cdot \boldsymbol{r} + \boldsymbol{z} \cdot \boldsymbol{s} + \log \lambda_{G;\phi}(\boldsymbol{x}, \boldsymbol{z})\}.$$

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Lemma

Set

$$\mathcal{G}([0,d),\delta) = \sup_{Q \in \mathcal{M}(G\times G;\varphi,[0,d)\times[0,d)\times[0,\delta))} H_Q(Y|X)$$

Then $|\mathcal{G}([0,d),\delta)| = \lim_{n\to\infty} \frac{1}{n} \log |B|$.

Spectral Radius

- For a function $f : E_G \to \mathbb{R}^k$, let $A_{G;f}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^k$, be the matrix function indexed by the states of G with entries

$$[A_{G;f}(\boldsymbol{x})]_{u,v} = \begin{cases} 2^{-\boldsymbol{x}\cdot f((u,v))} & \text{if } (u,v) \in E_G \\ 0 & \text{otherwise.} \end{cases}$$

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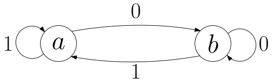
Example



$$A_{G;f}(x) = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \left[\begin{array}{cc} 1 & 2^{-x} \\ 2^{-x} & 1 \end{array} \right] \end{array}$$

$$\lambda_{G;f}(x) = 2^{-x} + 1$$

Lower Bound II: Markov Type

Markov Type Lower Bound

$$C(d,\delta) \geq \sup_{p\in[0,d]} \left\{ 2\,h(p) - \inf_{\substack{x\in\mathbb{R}\\ z\in\mathbb{R}_{\geq 0}}} \{2px + \delta z + \log \lambda_{G\times G;\varphi'}(x,z)\} \right\}$$

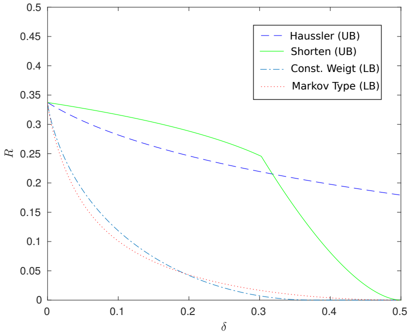
where

$$\lambda_{G\times G;\varphi'}(x,z) = \frac{1}{2}\Big((4^{-x}+1)(2^{-z}+1)+$$
$$\sqrt{(4^{-x}+1)^2 4^{-z} - 2(16^{-x} - 6\cdot 4^{-x}+1)2^{-z} + (4^{-x}+1)^2}\Big).$$

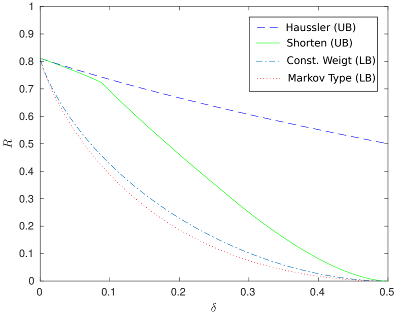
Proof. Follows the same line of reasoning as in [Marcus and Roth (1992)] where improved GV bound for constrained systems has been developed, based on stationary Markov chain analysis.

EXAMPLES: $\mathrm{VCD}(\mathcal{C}) = dn$

$d = \frac{1}{16}$

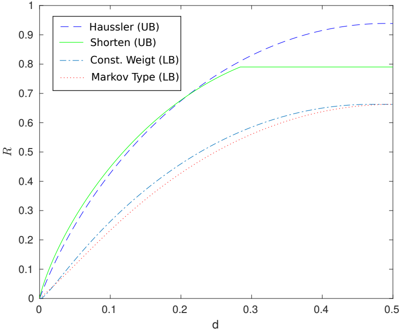


$d = \frac{1}{4}$

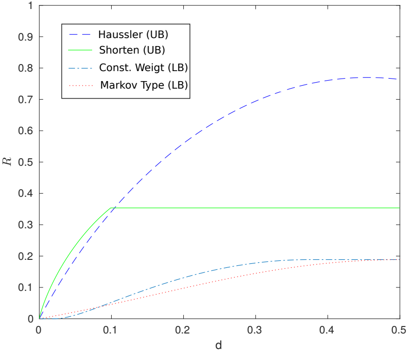


EXAMPLES: $\mathrm{dist}(\mathcal{C}) = \delta n$

$\delta = \frac{1}{16}$



$\delta = \frac{1}{4}$



FURTHER RESEARCH

- Upper bound
  - Minimum distance for prefixes in projection
  - Smart shifting?
  - VCD constraint in linear programming?
- Lower bound
  - Asymmetric Markov type
  - Other sets S
- Applications

Further Research

- Upper bound
  - Minimum distance for prefixes in projection
  - Smart shifting?
  - VCD constraint in linear programming?
- Lower bound
  - Asymmetric Markov type
  - Other sets S
- Applications

Thank You !