

ON THE VC-DIMENSION OF BINARY ERROR-CORRECTING CODES

Sihuang Hu

Joint work with Nir Weinberger and Ofer Shayevitz



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- **Fact:** The VC-dimension of the above code is 3.

TWO (EXTREMAL) EXAMPLES

- \mathcal{C} = Linear code of dimension k and with generator matrix

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Large Minimum Distance & Small VC-Dimension

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Examples

\mathcal{C}	Mini. Dist.	VCD
Linear codes	Large	Large
Hamming balls	Small	Small
Random codes	Large	Large

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- Rate $R \in [0, 1]$ is (δ, d) -achievable if $\exists \{\mathcal{C}_n\}_{n=1}^{\infty}$ such that

$$\text{dist}(\mathcal{C}_n) \geq \delta n$$

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- In this talk: two upper bounds + two lower bounds

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Corollary of Haussler's packing lemma (1995)

$$C(d, \delta) \leq \frac{2d}{\delta + 2d} \cdot h\left(\left\langle \frac{\delta + 2d}{2} \right\rangle\right)$$

Here $\langle a \rangle \stackrel{\text{def}}{=} \min\{a, \frac{1}{2}\}$.

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PF. The Probabilistic Method – Linearity of Expectation.

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$|\mathcal{Z}(\mathbf{u})|$: second MRRW bound.

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Upper bound via projection (H., Weinberger, Shayevitz' 17)

$$C(d, \delta) \leq \min_{0 \leq s \leq 1-2\delta} \left\{ s \cdot h \left(\left\langle \frac{d}{s} \right\rangle \right) + (1-s) \cdot R_{LP} \left(\frac{\delta}{1-s} \right) \right\}$$

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For $\mathcal{C} \subseteq \{0, 1\}^n$, there exists a Hamming ball of radius dn with at least

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codewords in it.

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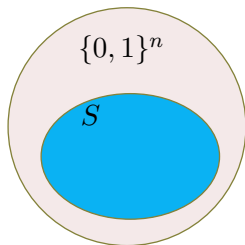
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- 3 We can do better via a direct approach!

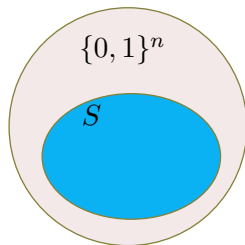
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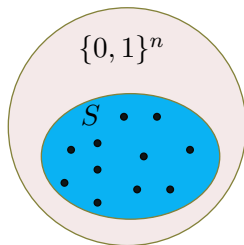
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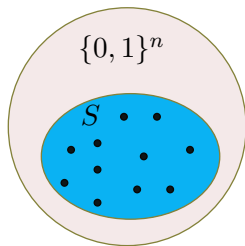
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- 4 Combine above bounds and evaluate $n \rightarrow \infty$ asymptotics.



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Proof: Suppose $\text{VCD}(\mathcal{C}) = dn$. Then $\exists \mathbf{x}, \mathbf{y} \in \mathcal{C}$ such that

$$\begin{aligned}\mathbf{x} &= (1 \ 1 \ 1 \ \cdots \ 1 \ x_{dn+1} \ \cdots \ x_n) \\ \mathbf{y} &= (0 \ 1 \ 1 \ \cdots \ 1 \ y_{dn+1} \ \cdots \ y_n)\end{aligned}$$

$$\delta n \leq \text{dist}(\mathbf{x}, \mathbf{y}) = 2wn - 2|\text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y})| \leq 2wn - 2(dn - 1)$$

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GV bound for constant weight codes

$$A(n, \delta n, wn) \geq \frac{\binom{n}{wn}}{\sum_{i=0}^{\delta n/2-1} \binom{wn}{i} \binom{n-wn}{i}}$$

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Constant-weight lower bound (H., Weinberger, Shayevitz' 17)

Let $d, \delta \in [0, \frac{1}{2}]$, and let $w = d + \frac{\delta}{2}$. Then

$$C(d, \delta) \geq \begin{cases} h(w) - \max_{0 \leq x \leq \delta/2} \left[w h\left(\frac{x}{w}\right) + (1-w)h\left(\frac{x}{1-w}\right) \right], & w < \frac{1}{2} \\ 1 - h(\delta), & \text{otherwise.} \end{cases}$$

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00110

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Generalized GV [Kolesnik, Krachkovsky'91]

There exists an $(S, M, \delta n)$ -code such that

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Remark. Improved by Gu and Fuja (1993), Tolhuizen (1997, implied by Turán's theorem).

STATIONARY MARKOV CHAINS ON GRAPHS

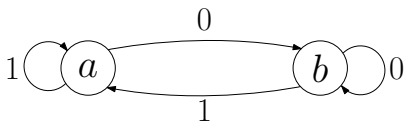
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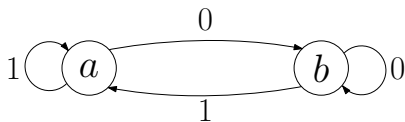
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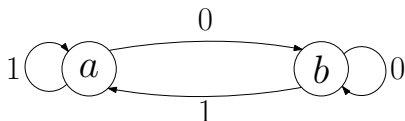
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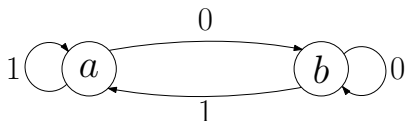


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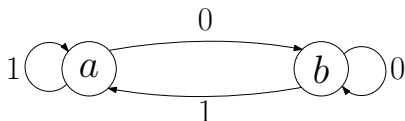


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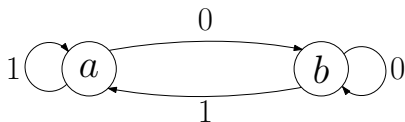


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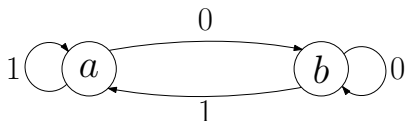


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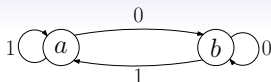
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Then the condition (II) amounts to saying that the marginal distributions of X and Y are equal.

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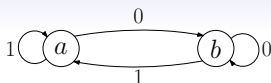


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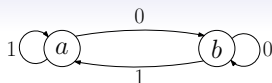
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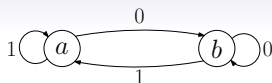
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$$\mathbf{E}_{P_\gamma}(f) = \sum_{e \in E_G} P_\gamma(e) f(e)$$

as the **empirical average** of f on the cycle γ .

SECOND-ORDER TYPES OF MARKOV CHAINS

Fact

For a cycle $\gamma = e_1 e_2 \cdots e_n \in \Gamma_n(G)$, the value $nE_{P_\gamma}(f) - f(e_1)$ is equal to the number of switches of the corresponding binary sequence $L_G(e_1)L_G(e_2)\cdots L_G(e_n)$.

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SECOND-ORDER TYPES OF MARKOV CHAINS

Lemma (Marcus, Roth'92)

Let G be a primitive graph and $f : E_G \rightarrow \mathbb{R}^k$ be a function on the edges of G . Let U be an open and nonempty subset of \mathbb{R}^k . Then

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This lemma is a consequence of well-known results on [second-order types of Markov chains](#), cf. Boza'71, Davisson, Longo, Sgarro'81, Natarajan'85, Csiszár, Cover, Choi'87, and Csiszár'98.

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Lemma (Convex duality principle, Marcus, Roth'92)

Let G be a graph and let $f : E_G \rightarrow \mathbb{R}^k$, $g : E_G \rightarrow \mathbb{R}^l$ be functions on the edges of G . Set $\phi = [f, g] : E_G \rightarrow \mathbb{R}^{k+l}$. Then for any $\mathbf{r} \in \mathbb{R}^k$ and $\mathbf{s} \in \mathbb{R}^l$,

$$\sup_{\substack{P \in \mathcal{M}(G): \\ \mathbf{E}_P(f) = \mathbf{r} \\ \mathbf{E}_P(g) \leq \mathbf{s}}} H_P(Y|X) = \inf_{\substack{\mathbf{x} \in \mathbb{R}^k \\ \mathbf{z} \in \mathbb{R}_{\geq 0}^l}} \{ \mathbf{x} \cdot \mathbf{r} + \mathbf{z} \cdot \mathbf{s} + \log \lambda_{G; \phi}(\mathbf{x}, \mathbf{z}) \}.$$

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Lemma

Set

$$\mathcal{G}([0, d], \delta) = \sup_{Q \in \mathcal{M}(G \times G; \varphi, [0, d] \times [0, d] \times [0, \delta])} H_Q(Y|X)$$

Then $|\mathcal{G}([0, d], \delta)| = \lim_{n \rightarrow \infty} \frac{1}{n} \log |B|$.

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- For a function $f : E_G \rightarrow \mathbb{R}^k$, let $A_{G;f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^k$, be the matrix function indexed by the states of G with entries

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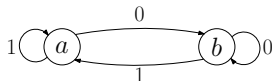
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Example



$$A_{G;f}(x) = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 2^{-x} \\ 2^{-x} & 1 \end{bmatrix} \end{matrix}$$

$$\lambda_{G;f}(x) = 2^{-x} + 1$$

LOWER BOUND II: MARKOV TYPE

Markov Type Lower Bound

$$C(d, \delta) \geq \sup_{p \in [0, d]} \left\{ 2h(p) - \inf_{\substack{x \in \mathbb{R} \\ z \in \mathbb{R}_{\geq 0}}} \{2px + \delta z + \log \lambda_{G \times G; \varphi'}(x, z)\} \right\}$$

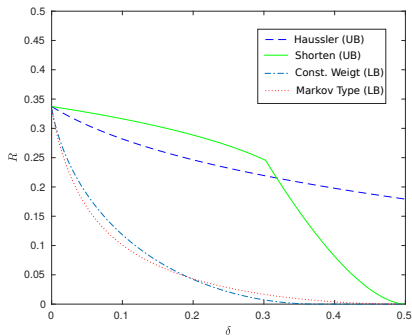
where

$$\lambda_{G \times G; \varphi'}(x, z) = \frac{1}{2} \left((4^{-x} + 1)(2^{-z} + 1) + \sqrt{(4^{-x} + 1)^2 4^{-z} - 2(16^{-x} - 6 \cdot 4^{-x} + 1)2^{-z} + (4^{-x} + 1)^2} \right).$$

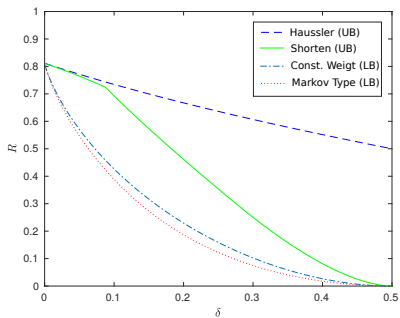
Proof. Follows the same line of reasoning as in [Marcus and Roth (1992)] where improved GV bound for constrained systems has been developed, based on stationary Markov chain analysis.

EXAMPLES: $VCD(\mathcal{C}) = dn$

$$d = \frac{1}{16}$$

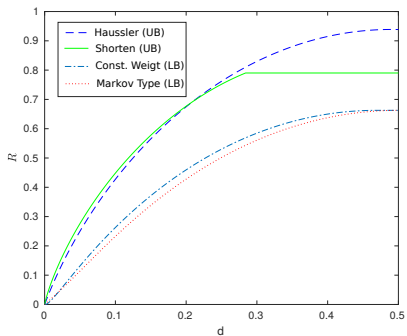


$$d = \frac{1}{4}$$

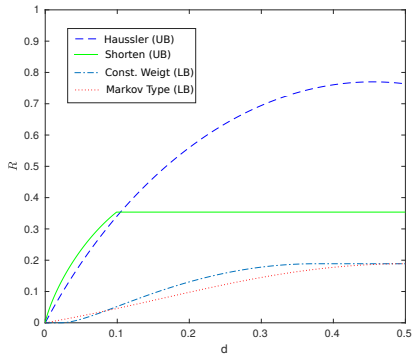


EXAMPLES: $\text{dist}(\mathcal{C}) = \delta n$

$$\delta = \frac{1}{16}$$



$$\delta = \frac{1}{4}$$



FURTHER RESEARCH

- Upper bound
 - Minimum distance for prefixes in projection
 - Smart shifting?
 - VCD constraint in linear programming?
- Lower bound
 - Asymmetric Markov type
 - Other sets S
- Applications

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Thank You !