A Brief Introduction to Lattice Coding Theory (in two parts)

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Introduction

• What is a lattice?
• Why study lattices?
• Real-world applications of lattices
Definition 1  An $n$-dimensional lattice $\Lambda$ is a discrete additive subgroup of $\mathbb{R}^n$. 

\begin{equation}
\Lambda = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{g}_i, \alpha_i \in \mathbb{Z} \}
\end{equation}
Lattice Definition

Definition 1  An $n$-dimensional lattice $\Lambda$ is a discrete additive subgroup of $\mathbb{R}^n$.

Vector addition in $\mathbb{R}^n$:

\[
\begin{align*}
\mathbf{x} &= [x_1, \ldots, x_n] \\
\mathbf{y} &= [y_1, \ldots, y_n] \\
\mathbf{x} + \mathbf{y} &= [x_1 + y_1, \ldots, x_n + y_n]
\end{align*}
\]

Group properties:

- has identity
- has inverse
- associative
- closure
- (commutative)
Lattices in $\mathbb{R}^2$
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Lattices in $\mathbb{R}^2$
Lattices in $\mathbb{R}^2$
Lattices Beyond $n=2$ dimensions

$n = 2$

$n = 3$

$n = 4, 5, 6, \ldots$

Figure 1.5: An $n=2$ Euclidean-space code with $M=16$ points.
Properties of lattices:

- Fundamental regions
- Minimum distance and coding gain
- Lattice transformations — scaling, rotation, reflections
- Lattice cosets
- Generator matrix, check matrix
- Quantization
- Lattice codes
Why Study Lattices?

- Lattices are error-correcting codes defined over the real numbers
- Lattices use the same real-number algebra as the physical world
  - physical-layer network coding or compute-forward
- Near-ideal codes for the AWGN channel
- Lossy source coding
- Lattices are fun
Communications on AWGN Channel

Encoder 
\( x \in C \) 

\[ \text{z} \] 

Decoder

\[ \text{AWGN} \sim \mathcal{N}(0, \sigma^2) \]
Variance = 10
N=16
Electromagnetic signals have linear superposition

- Codes over the real numbers are natural
- Lattices have a group structure — physical layer network coding
2 Users Signals Superimpose Linearly

\[ s_2(t) = A_2 \cos(\omega t + \phi_2) \]
\[ s_1(t) = A_1 \cos(\omega t + \phi_1) \]

Transmitted signals add

Signal real/imaginary \( A_i \exp(\phi\sqrt{-1}) \)
Lattice Compute and Forward

\[ u_1 = h_1 x_1 + h_2 x_2 \]

- relay only wants a linear combination
- transmitters do not need to know \( h_1, h_2 \)

Fading coefficients
Lattice Compute and Forward

\[ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

Invert this matrix, recover \( x_1, x_2 \)
AWGN Channel, Power Constraint

Input power constraint \( P \):
\[
\frac{1}{n} \sum_{i=1}^{n} x_i^2 \leq P
\]

Capacity is:
\[
R < C = \frac{1}{2} \log(1 + \frac{P}{\sigma^2})
\]
Input Distribution of a Codebook

SHANNON  Spherical codebook

\[ \frac{1}{n} \sum_{i=1}^{n} x_i^2 \leq P \]

LATTICE  Nested Lattice Code

Gaussian input distribution

\[ n \rightarrow \infty \]
Practical Applications of Lattices

- V.34 telephone cmodems at 33.6 kbits uses D4 lattice.
- Lossy source coding ITU-T 729.1 speech-coding Gosset lattice E8

Input

\[ y = (y_1, y_2, \ldots, y_n) \]

speech, image, etc.

Lattice \( \Lambda \) Quantizer

Output

\[ x \in \Lambda \text{ closest to } y \]
Lattices Are Fun
Study of Regular Division of the Plane with Reptiles (1939) by M. C. Escher
Systematic Way to Form Tessellations

Efficient Arrangement of Spheres

In two dimensions

Rectangular

Hexagonal
is most efficient
Efficient Arrangement in Space

Rectangular

Hexagonal is most efficient
All the Elegance of Lattices

G. David Forney,
Lecture notes for
Principles of Digital Communications II
Course at MIT

Google “David Forney lecture notes”
Ram Zamir, 
*Lattice Coding for Signals and Networks*
Cambridge University Press
September 2014