Association Schemes and Injection Codes

Nathan Lindzey (Technion)
Joint work w/ Peter Dukes and Ferdinand Ihringer

December 18, 2022

Association Schemes

The Injection Scheme

Injection Codes

Table of Contents

Association Schemes

A *association scheme* of rank d is a finite set X together with a set of $d+1$ nonempty $|X| \times |X|$ binary matrices $\mathcal{A} = \{A_i : 0 \leq i \leq d\}$ that satisfy the following conditions:

1. $A_0 = I$,
2. $A_0 + A_1 + \cdots A_d = J$ where J is the all-ones matrix,
3. $A_i \in \mathcal{A} \Rightarrow A_i^T \in \mathcal{A}$, and
4. $A_i A_j = A_j A_i = \sum_{i=0}^{d} p_{i,j}(k) A_k$ for all $0 \leq i, j, k \leq d$.

- The coefficients $p_{i,j}(k) \in \mathbb{N}$ are called the *intersection numbers*.
- We say $\mathcal{A}$ is *symmetric* if $A_i^\top = A_i$ for all i.
- Each *associate* A_i for $1 \leq i \leq d$ is a regular simple graph of *valency* v_i.
- The matrix algebra $\mathfrak{A} := \mathrm{Span}\{A_0, A_1, \ldots, A_d\}$ is called the *Bose–Mesner algebra*.

Idempotents

Given an association scheme $\mathcal{A}$, there exists a canonical dual basis of *primitive idempotents* $E_0, E_1, \ldots, E_d \succeq 0$ that satisfy the following conditions:

1. $E_0 = \frac{1}{|X|} J$,
2. $E_0 + E_1 + \cdots E_d = I$,
3. $E_i E_j = \delta_{i,j} E_i$ for all $0 \leq i, j \leq d$, and
4. $E_i \circ E_j = \sum_{i=0}^{d} q_{i,j}(k) E_k$ where $\circ$ denotes the *Hadamard* (entrywise) product.

- The coefficients $q_{i,j}(k) \in \mathbb{R}_{\geq 0}$ are called the *Krein parameters*.
- Each primitive idempotent E_i is the orthogonal projector onto the i-eigenspace.
- Moreover, we call $m_i := \operatorname{Tr} E_i$ the *multiplicity* of the i-eigenspace (i.e., dimension).

Eigenmatrices

Since $A_0, A_1, \cdots, A_d$ and $E_0, E_1, \cdots, E_d$ are bases, there exist $p_i(j), q_j(i)$ such that

$$A_i = \sum_{j=0}^{d} p_i(j) E_j \quad \text{and} \quad E_j = \frac{1}{|X|} \sum_{i=0}^{d} q_j(i) A_i \quad \text{for all } 0 \leq i, j \leq d.$$

There exist $(d+1) \times (d+1)$ change-of-basis matrices P, Q defined such that

$$P_{j,i} := p_i(j) \quad \text{and} \quad Q_{i,j} := q_j(i)$$

called the *first* and *second eigenmatrices* of $\mathcal{A}$, that is, $PQ = |X|I = QP$.

- The $p_i(j)$'s are the *eigenvalues* of $\mathcal{A}$ (i.e., $A_i E_j = p_i(j) E_j$).
- The $q_j(i)$'s are the *dual eigenvalues* of $\mathcal{A}$ (i.e., $A_i \circ E_j = q_j(i) A_i$).

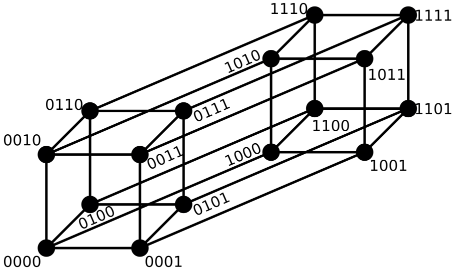
The Hamming Scheme

Let $X = \{1, 2, \cdots, q\}^d$. Let $d(x, y)$ denote the *Hamming distance* between $x, y \in X$.

For all $0 \leq i \leq d$, define the $|X| \times |X|$ matrix A_i such that

$$A_i[x, y] := \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $\mathcal{H}_{d,q} = \{A_i : 0 \leq i \leq d\}$ form *the Hamming (association) scheme.*



$\underline{A_1 \in \mathcal{H}_{4,2}}$

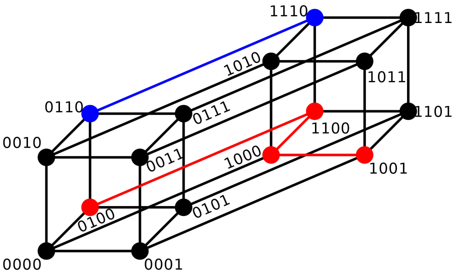
The Hamming Scheme

Let $X = \{1, 2, \cdots, q\}^d$. Let $d(x, y)$ denote the *Hamming distance* between $x, y \in X$.

For all $0 \leq i \leq d$, define the $|X| \times |X|$ matrix A_i such that

$$A_i[x, y] := \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $\mathcal{H}_{d,q} = \{A_i : 0 \leq i \leq d\}$ form *the Hamming (association) scheme*.



$A_1 \in \mathcal{H}_{4,2}$

The Hamming Scheme

The dual eigenvalues of $\mathcal{H}_{d,q}$ are given by the classical *Krawtchouk polynomials*:

$$\phi_k(x) = \sum_{j=1}^{k} \binom{x}{j} \binom{d-x}{k-j} (-1)^j (q-1)^{k-j}.$$

In particular, $Q_{i,k} = \phi_k(i)$. For $d = 3$ and $q = 2$, we have

$$Q = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{bmatrix}.$$

Conversely, the eigenvalues of $\mathcal{H}_{d,q}$ are given by the *dual Krawtchouk polynomials*.

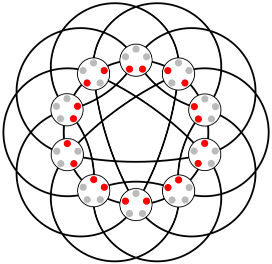
The Johnson Scheme

Let $X = \binom{[n]}{k}$ be the collection of k-sets of $[n] := \{1, 2, \cdots, n\}$.

For all $0 \leq i \leq k$, define the $|X| \times |X|$ matrix A_i such that

$$A_i[x, y] := \begin{cases} 1 & \text{if } |x \cap y| = k - i \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $\mathcal{J}_{n,k} = \{A_i : 0 \leq i \leq d\}$ form *the Johnson (association) scheme*.



$A_1 \in \mathcal{J}_{5,2}$

The Johnson Scheme

The eigenvalues of $\mathcal{J}_{n,k}$ are given by the classical *Eberlein polynomials*:

$$E_\ell(x) = \sum_{j=0}^{\ell} (-1)^j \binom{x}{j} \binom{k-x}{\ell-j} \binom{n-k-x}{\ell-j}.$$

In particular, $P_{\ell,i} = E_\ell(i)$.

Conversely, the dual eigenvalues of $\mathcal{J}_{n,k}$ are given by the *dual Eberlein polynomials*.

Delsarte's Approach

Let G_i be the v_i-regular graph associated to A_i. Let $Y \subseteq X$. Let $y \in \mathbb{R}^{d+1}$ such that

$$y_i = \frac{2|E(G_i[Y])|}{|Y|} \left(= \frac{1_Y A_i 1_Y^\top}{|Y|} \right) \quad \text{for all} \quad 0 \le i \le d.$$

- $y_0 = 1$,
- $y_i \ge 0$ for all i, and
- $y_0 + y_1 + \cdots + y_d = |Y|$.

We call y *the distribution vector* of Y.

Placing more structure on the set Y can give more linear constraints on y.

Delsarte's Approach

In the Hamming scheme, pick $Y \subseteq X$ to be the codewords of a certain type of code.

For example, if we want Y to be codewords of distance r, then we must have

$$y_1 = y_2 = \cdots = y_{r-1} = 0.$$

Delsarte's observation is that second eigenmatrix Q places $(d+1)$ linear inequalities that must hold for any distribution vector y of a set Y in an association scheme, i.e.,

$$yQ \geq 0.$$

The Delsarte LP

The linear program (DLP) below is *Delsarte's LP* and holds for any association scheme.

(DLP) maximize $\sum_{i=0}^{d} y_i$ subject to

- $y_0 = 1$
- $yQ \geq 0$,
- $y_i = 0$ for all $1 \leq i < r$,
- $y_i \geq 0$ for all $r \leq i \leq d$.

Since $\sum_{i=0}^{d} y_i = |Y|$, this gives an *upper bound* on the size of *any* distance-r code.

Delsarte proved (DLP) $\leq \frac{q^n}{\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k}(q-1)^k}$, i.e., the Hamming/sphere-packing bound.

DLP $(n = d, q = 2, \delta = r)$

n	δ	Hamming Bound	Linear Programming Bound
11	3	170.7	170.7
11	5	30.6	24
11	7	8.8	4
12	3	315.1	292.6
12	5	51.9	40
12	7	13.7	5.3
13	3	585.1	512
13	5	89.0	64
13	7	21.7	8
14	3	1092.3	1024
14	5	154.6	128
14	7	34.9	16
15	3	2048	2048
15	5	270.8	256
15	7	56.9	32

Better Bounds?

(DLP) works within the Bose–Mesner algebra $\mathfrak{A}$ — a *commutative* matrix algebra.

There is a larger matrix algebra coming from an association scheme $\mathcal{A}$ called the *Terwilliger algebra* T of $\mathcal{A}$ — a typically *non-commutative* matrix algebra.

Also, $\mathfrak{A} \subseteq \mathrm{T}$, so working harder (e.g., solving SDPs) over T may give better bounds.

Schrijver used the Terwilliger algebra of $\mathcal{H}_{d,2}$ and $\mathcal{J}_{n,k}$ to improve the state-of-the-art bounds on binary codes and constant-weight binary codes.

Gijswijt et al. (arXiv:1005.4959) extend these results using *SDP symmetry reduction*.

Asymptotics? Linear Codes?

Assume $q = 2$ and let $\delta \in (0,1)$. An asymptotic measure of rate estimated by DLP is

$$\limsup_{n\to\infty} \frac{\log_2 \ (DLP)_{n,\lfloor \delta n \rfloor}}{n} \leq H(1/2 - \sqrt{\delta(1-\delta)}).$$

The latter bound is due to McEliece, Rodemich, Rumsey, and Welch (see also Navon and Samorodnitsky), and remains best for $\delta \geq 0.273$.

Asymptotics? Linear Codes?

Assume $q = 2$ and let $\delta \in (0,1)$. An asymptotic measure of rate estimated by DLP is

$$\limsup_{n\to\infty} \frac{\log_2 \ (DLP)_{n,\lfloor \delta n \rfloor}}{n} \leq H(1/2 - \sqrt{\delta(1-\delta)}).$$

The latter bound is due to McEliece, Rodemich, Rumsey, and Welch (see also Navon and Samorodnitsky), and remains best for $\delta \geq 0.273$.

Recall that (DLP) gives bounds on *any* code.

Can one extend (DLP) to prove bounds on the sizes of linear codes?

- Loyfer and Linial (arXiv:2206.09211)
- Coregliano, Jeronimo, and Jones (arXiv:2211.01248)

Delsarte's Approach

Delsarte's approach gives a unified framework that unites coding and design theory.

$$\underbrace{\text{codes}}_{\mathcal{H}_{d,q}} \longleftrightarrow \textbf{cliques in association schemes} \longleftrightarrow \underbrace{\text{combinatorial designs}}_{\text{other schemes}}$$

We call these designs *codes* if they arise from a communication problem.

Many association schemes:

- Abelian groups: $\mathbb{Z}_p$, $\mathbb{Z}_2^n$, etc..
- non-Abelian groups: S_n, A_n, $GL(n, q)$, etc..
- Constant weight codes over non-binary alphabets (non-binary Johnson scheme)
- Finite Geometries
- Rooted d-ary trees
- Perfect Matchings of K_{2n}
- $\vdots$

Permutation Codes

Let S_n be the *symmetric group*, i.e., the set of all permutations of $[n]$.

Two permutations $\pi, \sigma \in S_n$ *agree* on the symbol $i \in [n]$ if $\pi(i) = \sigma(i)$.

We say $\pi, \sigma \in S_n$ have *(Hamming) distance d* if they agree on exactly $n - d$ symbols.

Clearly any two distinct permutations have Hamming distance at least 2.

A distance-d *permutation code* is a set $C \subseteq S_n$ such that $d(\sigma, \pi) \geq d$ for all $\sigma, \pi \in C$.

Let $M(n, d)$ denote the maximum size of a distance-d permutation code of S_n.

The Permutation Scheme

The cycle types of permutations of S_n correspond to *integer partitions* $\lambda \vdash n$, e.g.,

$$(1\ 2\ 3\ 4)(6\ 7\ 5)(8\ 9) \in S_{10} \quad \text{has cycle type} \quad (4,3,2,1) \vdash 10.$$

Let A_λ be the $n! \times n!$ matrix defined such that

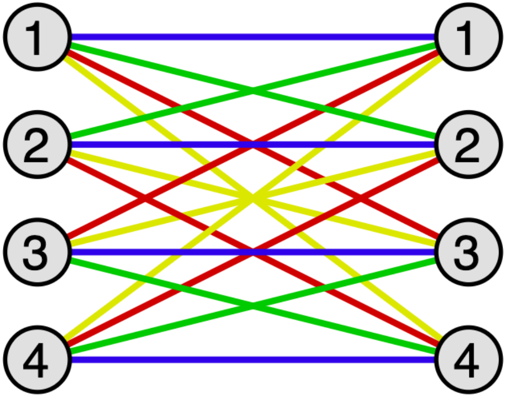
$$A_\lambda[\pi, \sigma] = \begin{cases} 1 & \text{the cycle type of } \sigma\pi^{-1} \text{ is } \lambda; \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $\mathcal{S}_n = \{A_\lambda\}_{\lambda \vdash n}$ form the *permutation (association) scheme* of order n.

- The valencies v_λ are the sizes of the λ-conjugacy classes C_λ of S_n
- The E_λ's are the $\perp$-projections onto the λ-isotypic components of $\mathbb{C}S_n$.
- The first eigenmatrix P of $\mathcal{S}_n$ is essentially the group character table of S_n.
- In (DLP), for distance-d codes we set $y_\lambda = 0$ if λ has more than $n - d$ 1's.

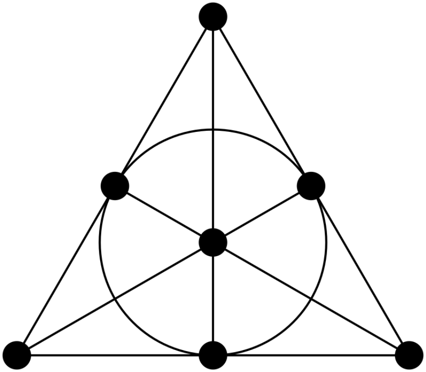
$M(n, n) = n$

A distance-n permutation code is a well-known design: *the latin square* of order n.



$M(n, n-1) = \ ?$

A largest distance-$(n-1)$ permutation code is a design: *the projective plane* of order n.



Infamous open question (even just for $n = 12$).

An Easier Scheme?

Difficult to make progress on permutation codes for arbitrary n and d.

Is there a scheme that somehow lies "in between" the Hamming/Johnson schemes and the permutation scheme?

$$\{\mathcal{H}_{d,q}, \mathcal{J}_{n,k}\} \quad \longleftrightarrow \quad ?? \quad \longleftrightarrow \quad \mathcal{S}_n.$$

Table of Contents

Injections

For all $k \leq n$, let $S_{k,n}$ be the set of all injective maps $\sigma : [k] \to [n]$.

$$|S_{k,n}| = (n)_k := n(n-1)\cdots(n-k+1)$$

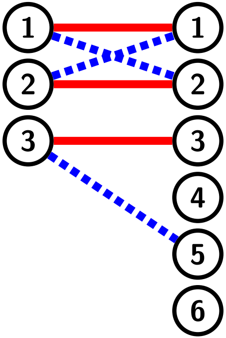
Injections

For all $k \leq n$, let $S_{k,n}$ be the set of all injective maps $\sigma : [k] \to [n]$.

$$|S_{k,n}| = (n)_k := n(n-1)\cdots(n-k+1)$$

Graphically, they are maximum matchings of $K_{k,n}$.

Example: $e = 123$, $\sigma = 215$.

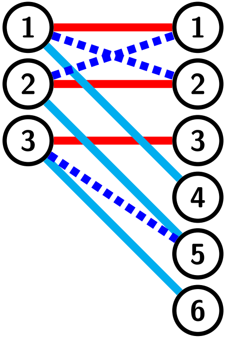


Cycle-Path Type

Let $e = 12 \cdots k \in S_{k,n}$ be the *identity injection* and $\sigma \in S_{k,n}$.

$$e \cup \sigma \cong \text{disjoint even-length cycles and even-length paths.}$$

Example: $e = ()$, $\sigma = (1,2)(3,5]$, and $\sigma' = (1,4](2,5](3,6]$.



Cycle-Path Type

Let $e = 12 \cdots k \in S_{k,n}$ be the *identity injection* and $\sigma \in S_{k,n}$.

Cycle-Path Type

Let $e = 12 \cdots k \in S_{k,n}$ be the *identity injection* and $\sigma \in S_{k,n}$.

Let λ and ρ be integer partitions that record the sizes of the cycle components and path components respectively.

Cycle-Path Type

Let $e = 12 \cdots k \in S_{k,n}$ be the *identity injection* and $\sigma \in S_{k,n}$.

Let λ and ρ be integer partitions that record the sizes of the cycle components and path components respectively.

Let $|\lambda|$ denote the *size* of an integer partition, and let and $\ell(\rho)$ denote the *length* of an integer partition, i.e., the number of parts.

Cycle-Path Type

Let $e = 12 \cdots k \in S_{k,n}$ be the *identity injection* and $\sigma \in S_{k,n}$.

Let λ and ρ be integer partitions that record the sizes of the cycle components and path components respectively.

Let $|\lambda|$ denote the *size* of an integer partition, and let and $\ell(\rho)$ denote the *length* of an integer partition, i.e., the number of parts.

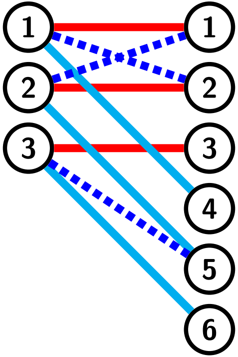
$$e \cup \sigma \text{ isomorphism types} \longleftrightarrow (\lambda|\rho) \text{ such that } |\lambda| + |\rho| = k \text{ and } \ell(\rho) \leq n - k$$

Let $C_{(\lambda|\rho)} \subseteq S_{k,n}$ be the set of all injections that have cycle-path type $(\lambda|\rho)$.

Cycle-Path Type

$$e \cup \sigma \text{ isomorphism types} \longleftrightarrow (\lambda, \rho) \text{ such that } |\lambda| + |\rho| = k \text{ and } \ell(\rho) \leq n - k$$

Example: $e \in C_{(1^3|\varnothing)}$, $\sigma \in C_{(2|1)}$, $\sigma' \in C_{(\varnothing|1^3)}$.



Cycle-Path Classes $C_{(\lambda|\rho)}$

Note that

$$S_{k,n} = \bigsqcup_{\substack{|\lambda|+|\rho|=k \\ \ell(\rho)\leq n-k}} C_{(\lambda|\rho)}.$$

Cycle-Path Classes $C_{(\lambda|\rho)}$

Note that

$$S_{k,n} = \bigsqcup_{\substack{|\lambda|+|\rho|=k \\ \ell(\rho)\leq n-k}} C_{(\lambda|\rho)}.$$

Let $\lambda = (0^{\ell_0}, 1^{\ell_1}, \cdots, k^{\ell_k})$ and $\rho = (0^{r_0}, 1^{r_1}, \cdots, k^{r_k})$. Then

$$|C_{(\lambda|\rho)}| = \frac{k!(n-k)!}{\prod_{i=0}^{k} i^{\ell_i} \ell_i! r_i!}.$$

When $k = n$, we recover the well-known formula for the size of a conjugacy class of S_n.

The Injection Scheme

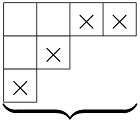
For all $k \leq n$, the *injection scheme* $\mathcal{S}_{k,n} := \{A_{(\lambda|\rho)}\}$ is the defined such that

$$A_{(\lambda|\rho)}[i,j] = \begin{cases} 1 & \text{if } i \cup j \cong (\lambda|\rho); \\ 0 & \text{otherwise.} \end{cases}$$

for all injections $i, j \in S_{k,n}$ and cycle-path types $(\lambda|\rho)$.

The $E_{(\lambda|\rho)}$'s are $\perp$-projectors onto certain *irreducible representations* of $S_k \times S_n$.

Example: $k = 3$, $n = 7$



$(\varnothing|2,1)$

The Injection Scheme

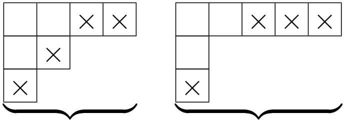
For all $k \leq n$, the *injection scheme* $\mathcal{S}_{k,n} := \{A_{(\lambda|\rho)}\}$ is the defined such that

$$A_{(\lambda|\rho)}[i,j] = \begin{cases} 1 & \text{if } i \cup j \cong (\lambda|\rho); \\ 0 & \text{otherwise.} \end{cases}$$

for all injections $i, j \in S_{k,n}$ and cycle-path types $(\lambda|\rho)$.

The $E_{(\lambda|\rho)}$'s are $\perp$-projectors onto certain *irreducible representations* of $S_k \times S_n$.

Example: $k = 3$, $n = 7$



$(\varnothing|2,1)$ $(1|2)$

The Injection Scheme

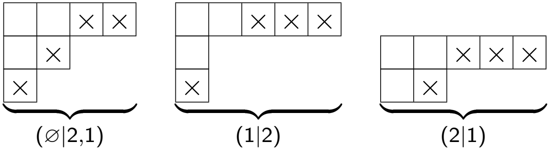
For all $k \leq n$, the *injection scheme* $\mathcal{S}_{k,n} := \{A_{(\lambda|\rho)}\}$ is the defined such that

$$A_{(\lambda|\rho)}[i,j] = \begin{cases} 1 & \text{if } i \cup j \cong (\lambda|\rho); \\ 0 & \text{otherwise.} \end{cases}$$

for all injections $i, j \in S_{k,n}$ and cycle-path types $(\lambda|\rho)$.

The $E_{(\lambda|\rho)}$'s are $\perp$-projectors onto certain *irreducible representations* of $S_k \times S_n$.

<u>Example:</u> $k = 3$, $n = 7$



The Injection Scheme

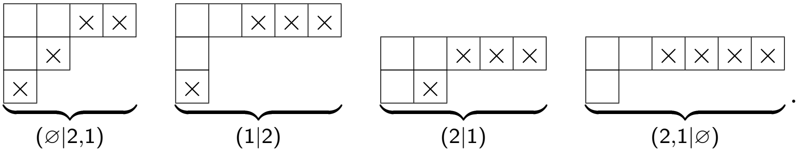
For all $k \leq n$, the *injection scheme* $\mathcal{S}_{k,n} := \{A_{(\lambda|\rho)}\}$ is the defined such that

$$A_{(\lambda|\rho)}[i,j] = \begin{cases} 1 & \text{if } i \cup j \cong (\lambda|\rho); \\ 0 & \text{otherwise.} \end{cases}$$

for all injections $i, j \in S_{k,n}$ and cycle-path types $(\lambda|\rho)$.

The $E_{(\lambda|\rho)}$'s are $\perp$-projectors onto certain *irreducible representations* of $S_k \times S_n$.

Example: $k = 3,\ n = 7$



The Injection Scheme

Some basic facts ...

- $v_{(\lambda|\rho)} = |C_{(\lambda|\rho)}|$
- $m_{(\lambda|\rho)}$ = the dimension of corresponding irreducible representation of $S_k \times S_n$.
- A simultaneous generalization of $\mathcal{S}_k$ and $\mathcal{J}_{n,k}$, i.e., $\mathcal{S}_k$ and $\mathcal{J}_{n,k}$ are *subschemes*.

The Injection Scheme

Some basic facts ...

- $v_{(\lambda|\rho)} = |C_{(\lambda|\rho)}|$
- $m_{(\lambda|\rho)} =$ the dimension of corresponding irreducible representation of $S_k \times S_n$.
- A simultaneous generalization of $\mathcal{S}_k$ and $\mathcal{J}_{n,k}$, i.e., $\mathcal{S}_k$ and $\mathcal{J}_{n,k}$ are *subschemes*.

... but many hard questions remain:

- Are there "good" formulas for computing the entries of the eigenmatrices P, Q?
  This is an old question of Persi Diaconis and Andy Greenhalgh.
- Are there "good" formulas for computing the Krein parameters $q_{i,j}(k)$?
  This turns out to have applications to Quantum Query Complexity.

The Injection Scheme

Some basic facts ...

- $v_{(\lambda|\rho)} = |C_{(\lambda|\rho)}|$
- $m_{(\lambda|\rho)}$ = the dimension of corresponding irreducible representation of $S_k \times S_n$.
- A simultaneous generalization of $\mathcal{S}_k$ and $\mathcal{J}_{n,k}$, i.e., $\mathcal{S}_k$ and $\mathcal{J}_{n,k}$ are *subschemes*.

... but many hard questions remain:

- Are there "good" formulas for computing the entries of the eigenmatrices P, Q? This is an old question of Persi Diaconis and Andy Greenhalgh.
- Are there "good" formulas for computing the Krein parameters $q_{i,j}(k)$? This turns out to have applications to Quantum Query Complexity.

We give a "decent" formula for computing the P matrix in the following work:

P. J. Dukes, F. Ihringer and N. Lindzey, *"On the Algebraic Combinatorics of Injections and its Applications to Injection Codes,"* in IEEE Transactions on Information Theory, vol. 66, no. 11, pp. 6898-6907, Nov. 2020.

Table of Contents

Injection Codes

Dukes '12

An *n-ary injection code* of length k and min. distance d is a set $C \subseteq S_{k,n}$ such that any two elements have Hamming distance $\geq d$.

For example, $31465 \in S_{5,6}$ and $21463 \in S_{5,6}$ have Hamming distance 2.

Let $M(n, k, d)$ be the maximum size of C.

$M(n, k, k) = n$ (Latin Rectangles).

Little known about $M(n, k, d)$ even for small k, n as Q-matrix was essentially unknown.

(DLP) for the Injection Scheme

Recall that $Q = (n)_k P^{-1}$ is the dual eigenmatrix (which we can now compute!).

(DLP) maximize $\sum_{(\lambda|\rho)} y_{(\lambda|\rho)}$ subject to

- $y_{(k|\varnothing)} = 1$
- $yQ \geq 0$,
- $y_{(\lambda|\rho)} = 0$ for all λ with more than $k - d$ 1's,
- $y_{(\lambda|\rho)} \geq 0$ for all remaining cycle-path types.

Upper bounds on $M(n,k,d)$ via (DLP)

Dukes, Ihringer, Lindzey '19

n	k	d	$M \leq$
7	6	4	199
8	6	3	1513
	7	4	1462
9	7	4	2846
	8	4	12096
		5	2417
10	7	3	27308
	8	4	26206
		5	5039
	9	4	92418
		5	19158
		6	4991
11	8	4	52646

n	k	d	$M \leq$
11	9	4	256682
		5	47073
	10	4	936332
		5	185560
		6	42068
12	8	3	602579
	9	4	584327
	10	4	2699260
		5	471981
	11	4	10241521
		5	1922527
		6	411090
13	9	4	1185053

Upper bounds on $M(n, k, d)$ via (DLP)

Dukes, Ihringer, Lindzey '19

n	k	d	$M \leq$
13	12	4	123235550
		5	23347599
		6	4687470
		7	910371
14	13	4	1621775700
		5	309490273
		6	58903464
		7	10510496
		8	2117618
15	14	4	23358981663
		5	4130012797
		6	804830167
		7	138132435
		8	24260981