The Generalized Covering Radii of Codes

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Motivation

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Examples

- Private information retrieval (PIR).

— see **Chor, Goldreich, Kushilevitz and Sudan**, *IEEE Trans. on Inform. Theory*, 1995.

- Partial-sum queries.

— see **Chazelle and Rosenberg**, *Proceedings of the fifth annual symposium on Computational geometry,*, 1989.

Linear Data Querying (Cont.)

Aspects in need of optimization

- Amount of storage at the server.
- The required bandwidth for the querying protocol.
- Access Complexity - the time required to access the elements of the database needed to compute the answer to a user query.

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Access Complexity

- The time required is proportional to the number of non-zero coefficients among $s_1, \ldots, s_m$.
- In the PIR scheme the coefficients are uniform i.i.d variables over $\mathbb{F}_q$ $\implies$ The expected number of non-zero coefficients is $\left(1 - \frac{1}{q}\right)m$.

Access Complexity and Storage

Reducing Access Complexity

- Design a set of linear combinations to be pre-computed and stored by the server.
- Instead of storing $\overline{x} = (x_1, \ldots, x_m)$ as is, the server stores $\overline{h}_1 \cdot \overline{x}, \ldots, \overline{h}_n \cdot \overline{x}$, where each $\overline{h}_i \in \mathbb{F}_q^m$ describes a linear combination.

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- Problem: The required storage amount is q^m instead of m.

Access Complexity-Storage-Latency Trade-Off

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- **Group Queries** - instead of answering one query at the time - accesses the combinations required for $t \geqslant 1$ queries together.

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- If we answer both queries together - by accessing $\overline{x}_1 + \overline{x}_2$ and $\overline{x}_3 - \overline{x}_2$, we may obtain

$$\overline{x}_3 + \overline{x}_1 = (\overline{x}_1 + \overline{x}_2) + (\overline{x}_3 - \overline{x}_2).$$

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Save 1 access to the database.

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Question

- Having the pre-computed linear combinations $\overline{h}_1 \cdot \overline{x}, \ldots, \overline{h}_n \cdot \overline{x}$, how many pre-computed combinations needs to be accessed in order to answer t linear queries together?

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- Namely, what is the minimal r such that any t coefficient vectors $\overline{s}_1, \ldots, \overline{s}_t$ can be computed as a linear combination of $r \leqslant m$ of $\overline{h}_1, \ldots, \overline{h}_n$. That is, there exits $\ell_1, \ldots, \ell_r$, such that $\overline{s}_j = \sum_{i=1}^{r} \alpha_i \overline{h}_{\ell_i}$? in that case, $\overline{s}_j \cdot \overline{x} = \sum_{i=1}^{r} \alpha_i (\overline{h}_{\ell_i} \cdot \overline{x})$

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- We define this number to be the t-th generalized covering radius of the $[n, n-m=k]_q$ linear code generated by the parity matrix H whose columns are $\overline{h}_1, \ldots, \overline{h}_n$.

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Definition

Let C be an $[n,k]_q$ linear code over $\mathbb{F}_q$, given by an $(n-k) \times n$ parity-check matrix $H \in \mathbb{F}_q^{(n-k)\times n}$ with columns $\overline{h}_1, \ldots, \overline{h}_n$. For every $t \in \mathbb{N}$ we define the *t-th generalized covering radius*, $R_t(C)$, to be the minimal integer $r \in \mathbb{N}$ such that for every set of vectors $S = \{\overline{s}_1, \ldots, \overline{s}_t\} \subseteq \mathbb{F}_q^{n-k}$, there exists $I \subseteq \mathbb{F}_q^{n-k}$, $|I| = r$ such that $S \subseteq \operatorname{span}\left\{\overline{h}_i \ : \ i \in I\right\}$.

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The Trade-Off as A Covering Problem

The smallest possible number of pre-computed combinations for answering a group of t queries accessing r database elements is lower bounded by the smallest possible length of a linear code with t-covering radius r and redundancy m over $\mathbb{F}_q$.

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Example

Let C be the binary $[2^m-1, 2^m-m-1, 3]$ Hamming code. The columns of the standard parity check matrix H are all the binary non-zero vectors of length m. What is $R_t(C)$ for $1 \leqslant t \leqslant m$?

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Given t non-zero vectors in $\mathbb{F}_2^m$, they are all columns of H. Thus, $R_t(C) \leqslant t$.

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Equivalent Definitions

t-Weights and t-distance

- For $t \in \mathbb{N}$ we define the t-weights on $\mathbb{F}_q^{t\times n}$. For a matrix $\mathbf{v} \in \mathbb{F}_q^{t\times n}$ with row vectors $\overline{v}_1, \ldots, \overline{v}_t$ we define

$$\mathrm{wt}^{(t)}(\mathbf{v}) \triangleq \left| \bigcup_{1\leqslant i\leqslant t} \mathrm{supp}(\overline{v}_i) \right|, \quad d^{(t)}(\mathbf{v}_1, \mathbf{v}_2) \triangleq \mathrm{wt}^{(t)}(\mathbf{v}_1 - \mathbf{v}_2).$$

- For a matrix $\mathbf{v} \in \mathbb{F}_q^{t\times n}$ we denote the ball or radius r centred in $\mathbf{v}$ (with respect to $d^{(t)}$) by $B_r^{(t)}(\mathbf{v})$, and it volume by $V_{r,n,q}^{(t)}$.
- If $C \subseteq \mathbb{F}_q^n$ is a linear code, we define C^t to be the set of all matrices in $\mathbb{F}_q^{t\times n}$ such that their rows belongs in C.

Equivalent Definitions(Cont.)

Geometric Definition

Let C be an $[n,k]_q$ code, we consider $C^t \subseteq \mathbb{F}_q^{t\times n}$. Then $R_t(C)$ is the regular covering radius of C^t with respect to $d^{(t)}$:

$$R_t(C) = \min\left\{ r \in \mathbb{N} \ : \ \bigcup_{\mathbf{c}\in C^t} B_r^{(t)}(\mathbf{c}) = \mathbb{F}_q^{t\times n} \right\}.$$

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Algebraic Definition

Let C be an $[n,k]_q$ linear code. Assume $G \in \mathbb{F}_q^{k\times n}$ is a generator matrix for C. Let C_t be the linear code over $\mathbb{F}_{q^t}$ generated by the same matrix G. That is, $C_t = \left\{ \overline{u}G \ : \ \overline{u} \in \mathbb{F}_{q^t}^k \right\}$. Then $R_t(C) = R_1(C_t)$.

The algebraic definition is related to the work of Helleseth on extension codes (1979).

Asymptotic Results

Bounds

Definition

Let $k_t(n, r, q)$ denote the smallest dimension of a linear code C over $\mathbb{F}_q$ with length n and t-covering radius $R_t(C) \leqslant r$.

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For a normalized covering radius $0 \leqslant \rho \leqslant 1$, the minimal rate achieving ρ is defined to be

$$\kappa_t(\rho, q) \triangleq \liminf_{n\to\infty} \frac{k_t(n, \rho n, q)}{n}.$$

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Lower Bound (Ball-Covering)

For $\rho \in \left(0, 1 - \frac{1}{q^t}\right)$,

$$\kappa_t(\rho, q) \geqslant 1 - H_{q^t}(\rho).$$

A Naive Upper Bound

Theorem (A Naive Upper Bound)

For $\rho \in [0, 1]$
$$\kappa_t(\rho, q) \leqslant \kappa_1\left(\frac{\rho}{t}, q\right) = 1 - H_q\left(\frac{\rho}{t}\right)$$

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Proof Sketch

We use a sub-additivity property:

For an $[n, k]_q$ linear code C and $t_1, t_2 \in \mathbb{N}$,

$$R_{t_1+t_2}(C) \leqslant R_{t_1}(C) + R_{t_2}(C).$$

As a consequence $R_t(C) \leqslant t \cdot R_1(C)$.

We combine with a result by Cohen and Frankl (1985)

$$\kappa_1(\rho, q) = 1 - H_q(\rho).$$

Improved Upper Bound for the binary case where $t = 2$

Theorem

For any $0 < \rho \leqslant 1$,

$$\kappa_2(\rho, 2) \leqslant \begin{cases} 1 - (4H_4(\rho) - f(\rho)) & 0 \leqslant \rho < \frac{3}{4}, \\ 0 & \frac{3}{4} \leqslant \rho \leqslant 1, \end{cases}$$

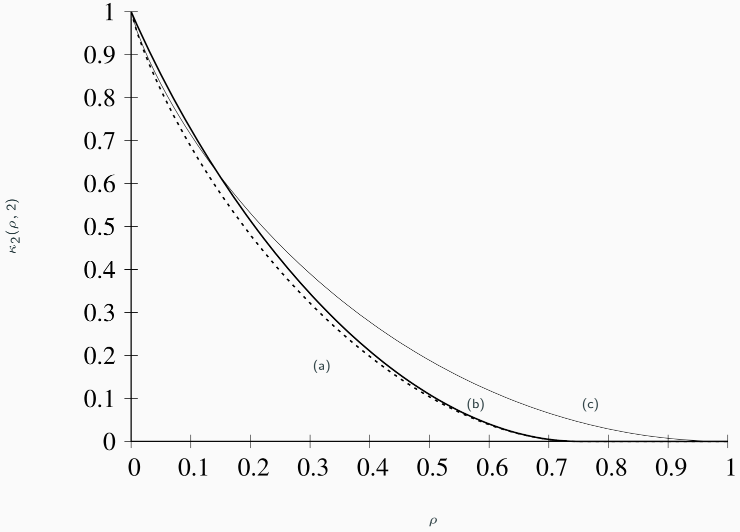
where

$$f(\rho) = \begin{cases} H_2(s(\rho)) + 2s(\rho) + 2(1 - s(\rho))H_2\left(\frac{\rho - s(\rho)}{1 - s(\rho)}\right) & 0 \leqslant \rho < \frac{3}{4}, \\ 3 & \frac{3}{4} \leqslant \rho \leqslant 1. \end{cases}$$

and

$$s(\rho) \triangleq \frac{1}{10}\left(1 + 8\rho - \sqrt{1 + 16\rho - 16\rho^2}\right).$$

A Comparison of The Bounds



A comparison of the bounds on $\kappa_2(\rho, 2)$: (a) the ball-covering lower bound, (b) the improved upper bound, and (c) the naive upper bound.

Improved Upper Bound for the binary case where $t = 2$

Proof Sketch

We use the probabilistic method.

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We use the probabilistic method. Assume $G \in \mathbb{F}_2^{k \times n}$ is a uniformly random matrix, and consider the code $C^2 = \left\{ \mathbf{u}G \ : \ \mathbf{u} \in \mathbb{F}_2^{2 \times k} \right\}$.

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$$X_{\mathbf{v}} \triangleq \sum_{\substack{\mathbf{u} \in \mathbb{F}_2^{2 \times k} \\ \text{rank}(\mathbf{u}) = 2}} \mathbb{I}_{\left\{ \mathbf{v} \in B_r^{(2)}(\mathbf{u}G) \right\}}.$$

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Using a careful analysis we can determine that

$$\begin{aligned} V_{r,n,2}^{(2)} \cdot 2^{2k-1-2n} < \mathrm{E}[X_{\mathbf{v}}] < V_{r,n,2}^{(2)} \cdot 2^{2k-2n}, \\ \mathrm{Var}(X_{\mathbf{v}}) \leqslant 7\,\mathrm{E}[X_{\mathbf{v}}] + 2^{3(k-n)+n(f(\rho)+o(1))}. \end{aligned}$$

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The code is then constructed in two stages:

1. We choose

$$k = \lceil n(1 - 4H_4(\rho) + f(\rho)) + \log_2(n) \rceil.$$

   We then show the code obtained when G has k rows already covers a large enough portion of the space.

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We then show the code obtained when G has k rows already covers a large enough portion of the space.
2. We then successively add $2\lceil \log_2(n) \rceil + 2$ rows to G that to guarantee the coverage of the entire space.

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- $r = \rho n$ represents the number of database elements we allow to access for answering 2 queries, $k = n - m$ represents the overhead.
- Further assume, that to handle the two queries we allow the server to access at most $\frac{1}{2}$ of its storage. Stated alternatively, the average access per query is a $\frac{1}{4}$ of the storage.

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- Consider the binary case, and assume we allow a latency of $t = 2$, namely, the server waits until two queries arrive and then handles them both.
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- A naive approach, using $\kappa_1(\frac{1}{4}, 2) \approx 0.19$, implies the storage may contain only 81% user information and 19% overhead.
- Since $\kappa_2(\frac{1}{2}, 2) \leqslant 0.11$, there exists a code allowing 89% of the server storage for user information and only 11% overhead.

A step towards the calculation of $\kappa_t(\rho, q)$

Conjecture

We conjecture that the lower bound meets the true value of $\kappa_t(\rho, q)$. That is, For any $0 \leqslant \rho \leqslant 1$,

$$\kappa_t(\rho, q) = \begin{cases} 1 - H_{q^t}(\rho) & 0 \leqslant \rho < 1 - \frac{1}{q^t}, \\ 0 & 1 - \frac{1}{q^t} \leqslant \rho \leqslant 1. \end{cases}$$

We know that it is true for $t = 1$(Cohen and Frankl 1985)

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General codes

We recall that by the geometric definition, R_t is defined for any code - $R_t(C)$ is the covering radius of $C^t \subseteq \mathbb{F}_q^{t \times n}$ with respect to $d^{(t)}$.

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We define $\hat{k}_t(n, r, q)$ to be the minimal dimension of a **general** code of length n with t-covering radius at most r and $\hat{\kappa}_t(\rho, q) \triangleq \liminf_{n \to \infty} \frac{\hat{k}_t(n, \rho n, q)}{n}$.

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We know that for any ρ and q, $\hat{\kappa}_1(\rho, q) = \kappa_1(\rho, q)$.

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Theorem

1. For $t = 2$ and any $q \in \mathbb{N}$ and $0 \leqslant \rho \leqslant 1$,

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2. In fact - if $\alpha_n(\varepsilon)$ fraction of codes with $R_2(C) \leqslant \rho n$ among codes of length n and size $\lfloor q^{n(\hat{\kappa}_2(\rho,q)+\varepsilon)} \rfloor$, then $\alpha_n(\varepsilon) \to 1$ as $n \to \infty$.

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The Next Step

We know that for any ρ and q, $\hat{\kappa}_1(\rho, q) = \kappa_1(\rho, q)$. The next would be to prove that:

$$\hat{\kappa}_2(\rho, q) = \kappa_2(\rho, q).$$

The Generalized Covering Radii of Some Known Codes

Let's start with the Hamming code

Theorem

Let C be the $[n = \frac{q^m-1}{q-1}, k = \frac{q^m-1}{q-1} - m, 3]$ Hamming code over $\mathbb{F}_q$. Then for all $1 \leqslant t \leqslant m$,

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Theorem

Let C be an $[n, k \geqslant 1, d \geqslant 3]$ linear code over $\mathbb{F}_q$. Then

$$R_1 < R_2 < \ldots < R_{n-k} = n - k,$$

if and only if C is the q-ary Hamming code.

Let's start with the Hamming code

Theorem

Let C be the $[n = \frac{q^m-1}{q-1}, k = \frac{q^m-1}{q-1} - m, 3]$ *Hamming code over* $\mathbb{F}_q$*. Then for all* $1 \leqslant t \leqslant m$,

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Theorem

Let C be an $[n, k \geqslant 1, d \geqslant 3]$ *linear code over* $\mathbb{F}_q$*. Then*

$$R_1 < R_2 < \ldots < R_{n-k} = n - k,$$

if and only if C is the q-ary Hamming code.

Proof.

Since $R_{n-k} = n - k$, we must have $R_1 = 1$. But a linear code with parameters $[n, k, d \geqslant 3]$ with covering radius $R_1 = 1$ is 1-perfect and it must be the q-ary Hamming code. □

MDS codes have a very narrow hierarchy

Theorem (Gabidulin and Kløve, *ITW*, 1998)

Let C be a q-ary $[n, k, n-k+1]$ MDS code. Then its (regular) covering radius is

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Corollary

Let C be a q-ary $[n, k, n-k+1]$ MDS code. Then for all $1 \leqslant t \leqslant n-k$,

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Proof.

Immediate given the monotonicity of the generalized covering radii. □

Reed-Muller codes might be interesting

Pros:

- They have a wide range of parameters.
- They have many equivalent definitions.
- They are useful in communications (attain capacity of symmetric and erasure channels, and are connected to locally decodable codes, probabilistic proof systems) and cryptography (connected to the study of Boolean functions and sequence design).

— see **Reeves and Pfister**, *arXiv*, 2021, **Kudekar et al.**, *IEEE Trans. on Inform. Theory*, 2017, **Yekhanin**, *Now*, 2012, **Abbe et al.**, *IEEE Trans. on Inform. Theory*, 2015, **Kurosawa et al.**, *IEEE Trans. on Inform. Theory*, 2004, **Schmidt**, *IEEE Trans. on Inform. Theory*, 2007

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Cons:

We don't even know their exact covering radius!

Here's a brief reminder

We shall find it convenient to use the following recursive construction of Reed-Muller codes:

Assume C_1 and C_2 are $[n, k_1]_q$ and $[n, k_2]_q$ codes, respectively. The $(u, u+v)$ construction uses C_1 and C_2 to produce a code

$$C = \{(\overline{u}, \overline{u}+\overline{v}) \mid \overline{u} \in C_1, \overline{v} \in C_2\}.$$

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- $\mathrm{RM}(0, m) \triangleq \left\{\overline{0}, \overline{1}\right\} \subseteq \mathbb{F}_2^{2^m}$

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- For $1 \leqslant r \leqslant m-1$, we define $\mathrm{RM}(r, m)$ to be the code produced by the $(u, u+v)$ construction using $\mathrm{RM}(r, m-1)$ and $\mathrm{RM}(r-1, m-1)$.

Results

Exact values

We have calculated exact t-th covering radii of $\mathrm{RM}(r, m)$ (denoted by $R_t(r, m)$) in some extreme cases:

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$R_t(0, m)$	$2^m - \lceil 2^{m-t} \rceil$	Repetitions code
$R_t(m-2, m)$	$\min\{t, m\} + 1$	Extended Hamming code
$R_t(m-1, m)$	1	Parity code
$R_t(m, m)$	0	Full code

Results (cont.)

Bounds

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   - $\alpha > \frac{1}{2}$ - the rate tends to 1 as $m \to \infty$.
4. RM(r, m) where $r = \frac{1}{2}m + \Theta(\sqrt{m})$ - the rate can converge to any constant number in $[0, 1]$.

Results (cont.)

A small reminder before we look at the bounds

It all started from linear data querying!

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- Given a fixed number of database elemnts - m, the dimension of the corresponding code is $k = n - m$, where n is number of pre-computed linear combinations. $\implies$ Since we want to reduce the number of pre-computed linear combinations - codes with low rate are desirable!

A summary of the bounds

$R_t(r,m)$	$\leqslant \left(1-\frac{1}{2^t}\right)2^m - \frac{\sqrt{2^t-1}}{2^t}(1+\sqrt{2})^{r-1}2^{m/2} + O(m^{r-2})$	
	$\geqslant \left(1-\frac{1}{2^t}\right)2^m - \frac{\sqrt{2t(2^t-1)\ln 2}}{2^t\sqrt{r!}}m^{r/2}2^{m/2}(1+o(1))$	
$R_t(m-s,m)$	$\leqslant \frac{t}{(s-2)!}m^{s-2} + O(m^{s-3})$	
	$\geqslant \frac{t}{(s-1)!}m^{s-2} + O(m^{s-3}\log(m))$	
$R_t(\alpha m,m)$	$\leqslant \left(1-\frac{1}{2^t}\right)2^m - \frac{\sqrt{2^t-1}}{2^t(2+\sqrt{2})}2^{m\left(\frac{1}{2}+\alpha\log_2(1+\sqrt{2})\right)}(1+o(1))$	$0<\alpha<1-\frac{1}{\sqrt{2}}$
	$\leqslant \left(1-\frac{1}{2^t}\right)2^m - \frac{\sqrt{2^t-1}}{2^t}\cdot\frac{1}{\sqrt{8m\alpha(1-\alpha)}}\cdot 2^{mH_2(\alpha)}$	$1-\frac{1}{\sqrt{2}}\leqslant\alpha<\frac{1}{2}$
	$\leqslant t\cdot 4^{H_2(\alpha)}\cdot 2^{mH_2(\alpha)}\cdot(1+o(1))$	$\frac{1}{2}<\alpha<1$
	$\geqslant \left(1-\frac{1}{2^t}\right)2^m - \frac{\sqrt{2t(2^t-1)\ln 2}}{2^t}\cdot 2^{\frac{m}{2}(1+H_2(\alpha))}\cdot(1+o(1))$	$0<\alpha<\frac{1}{2}$
	$\geqslant t\cdot\sqrt{\frac{1-\alpha}{8(\alpha m)^3}}\cdot 2^{mH_2(\alpha)}\cdot(1+o(1))$	$\frac{1}{2}<\alpha<1$
$R_t(r,m)$	$\leqslant \left(1-\frac{1}{2^t}\right)2^m - \frac{\sqrt{2^t-1}}{2^t}\frac{2^m}{\sqrt{\frac{1}{2}m\pi}}e^{-\frac{(m-2r)^2}{2m}}(1+o(1))$	$\sum_{i=0}^{r}\binom{m}{i}=\kappa 2^m$
	$\geqslant H_{2^t}^{-1}(1-\kappa)2^m(1+o(1))$	

Proofs Ideas - Lower Bounds

Algebraic definition of the generalized covering radius

Let C be an $[n, k]_q$ linear code. Assume $G \in \mathbb{F}_q^{k \times n}$ is a generator matrix for C. Let C_t be the linear code over $\mathbb{F}_{q^t}$ generated by the same matrix G. That is, $C_t = \left\{ \overline{u}G \ : \ \overline{u} \in \mathbb{F}_{q^t}^k \right\}$. Then $R_t(C) = R_1(C_t)$.

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Lemma - ball-covering argument

For an $[n,k]_q$ code C and

$$\log_q\left(V_{q,n,R_1(C)}\right) \geqslant n-k,$$

where $V_{q,n,r}$ denotes the volume of the Hamming ball of radius r in $\mathbb{F}_q^n$.

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where $V_{q,n,r}$ denotes the volume of the Hamming ball of radius r in $\mathbb{F}_q^n$.

Corollary

Applying on the ball-covering argument on C_t:

$$\log_{q^t}\left(V_{q^t,n,R_t(C)}\right) = \log_{q^t}\left(V_{q^t,n,R_1(C_t)}\right) \geqslant n-k.$$

Proofs Ideas - Upper Bounds

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if a code C is produced using the $(u, u+v)$ construction with C_1 and C_2, then

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Corollary

Since Reed-Muller codes are obtained from the $(u, u+v)$ construction - for any $m \geqslant 2$ and $1 \leqslant r \leqslant m-1$

$$R_t(r, m) \leqslant R_t(r-1, m-1) + R_t(r, m-1).$$

Strategy

We upper bound the extreme cases and recursively apply the $(u, u+v)$ bounds. By careful analysis, we obtain our bounds in the considered scenarios.

Let us go back to our motivating scenario

Assume a server stores linear combinations of the database items according to the columns of a parity-check matrix H of $\mathrm{RM}(r, m)$.

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After translating the problem to a coding theoretic one:

Goal

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Question

Can we find a solution that requires a distance that is no more than the upper bounds we presented on $R_t(r, m)$?

Yes, we can!

Main Idea

- Our bounds are based on subadditivity:

$$R_t(r, m) \leqslant R_t(r-1, m-1) + R_t(r, m-1).$$

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- The recursion is grounded in two base cases:
  1. $\mathrm{RM}(m, m)$ is the entire space, and hence the nearest codeword are the received matrix.
  2. $(\mathrm{RM}(1, m))^t$ is small enough to use a brute-force search to find the best codeword.

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  1. $\mathrm{RM}(m, m)$ is the entire space, and hence the nearest codeword are the received matrix.
  2. $(\mathrm{RM}(1, m))^t$ is small enough to use a brute-force search to find the best codeword.
- The time complexity: $O(t2^t(2^{t+1})^{m+1}(2^{t+1}-1)^{-r} + tm2^m)$.

The algorithm is simple

Algorithm 1: A t-covering algorithm for $\mathrm{RM}(r, m)$ with radius $U_t(r, m)$

Function `recursive`($\mathbf{v}$, r)

Input : $\mathbf{v} \in \mathbb{F}_2^{t \times 2^m}$, $r \in \mathbb{N}$, $1 \leqslant r \leqslant m$

`// Check edge cases`

if $r = m$ **then** **return** $\mathbf{v}$

if $r = 1$ **then** **return** $\mathrm{argmin}_{\mathbf{c} \in \mathrm{RM}(1,m)^t} d^{(t)}(\mathbf{v}, \mathbf{c})$

`// Use the` $(u, u + v)$ `recursion`

Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{F}_2^{t \times 2^{m-1}}$ s.t. $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$

$\mathbf{c}_1 \leftarrow \texttt{recursive}(\mathbf{v}_1, r)$

$\mathbf{c}_2 \leftarrow \texttt{recursive}(\mathbf{v}_2 - \mathbf{c}_1, r - 1)$

return $(\mathbf{c}_1, \mathbf{c}_1 + \mathbf{c}_2)$

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The generalized covering radius of linear codes is a completely new topic and many interesting directions for future research remain. For example:

- Computing the generalized covering radius for known families of codes.
- Improving the asymptotic bounds, and generalizing them.
- Constructing codes with a given t-covering radius.

Thank you for your attention!